King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 201

Major Exam 1

212

February 23, 2022

Net Time Allowed: 90 Minutes

MASTER VERSION

Key Solution

1. The parametric curve

$$x = \frac{1}{\sqrt{t+1}} \text{ and } y = \frac{t}{t+1}, \quad t > -1$$

represents:

(a) the right branch of the parabola $y = 1 - x^2$ excluding the vertex
(b) the parabola $y = x^2$ excluding the vertex
(c) the parabola $x = 1 - y^2$
(d) the upper branch of the parabola $x = 1 - y^2$ excluding the vertex
(e) the parabola $y = 1 - x^2$

$$x^2 = \frac{1}{t+1} \Rightarrow t = \frac{1-x^2}{x^2}$$

$$\Rightarrow y = \frac{t}{t+1} \Rightarrow \boxed{y = 1 - x^2} \quad x \neq 0$$

$$t > -1 \Rightarrow \boxed{x > 0} \rightarrow \text{right branch}$$

2. If the position of two particles at time t $(0 \le t \le 2\pi)$ is given by

$P_1: \; x_1 = 3 \sin t, \qquad y_1 = 2 \cos t$

$P_2: \; x_2 = -3 + \cos t, \qquad y_2 = 1 + \sin t$

Set $x_1 = x_2$ and $y_1 = y_2$

Then the collision point of the two particles is

(a) $(-3, 0)$
(b) $(0, 2)$
(c) $(-1, 2)$
(d) $(3, 2)$
(e) $(-1, -3)$

$3\sin t = -3 + \cos t$ — (1)

$2\cos t = 1 + \sin t$ — (2)

From (2): $3 \sin t = 6\cos t - 3$

Plug in (1): $6\cos t - 3 = -3 + \cos t$

$\Rightarrow \boxed{\cos t = 0} \Rightarrow t = \frac{\pi}{2}, 3\frac{\pi}{2}$

If $t = \frac{\pi}{2}$: eq^n (1) $\Rightarrow 3 = -3$ (false)

However: $\boxed{t = 3\frac{\pi}{2}}$ satisfies both (1) & (2)

$\Rightarrow t = 3\frac{\pi}{2} \Rightarrow$ Collision point is $(-3, 0)$

3. Consider the parametric curve

$$C:\ x = 1 + t^2\ ,\ y = \frac{1}{t^2}.$$

An equation of the tangent line to C at the point $P(2,1)$ is

(a) $y = -x + 3$

(b) $y = x + 2$

(c) $y = -x - 1$

(d) $y = x - 1$

(e) $y = x - 3$

$x = 2 \Rightarrow 1 + t^2 = 2 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

$\frac{\partial x}{\partial t} = 2t$ and $\frac{\partial y}{\partial t} = -\frac{2}{t^3}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{t^4}}$

$\left.\frac{dy}{dx}\right|_{t=\pm 1} = -1 \Rightarrow y - 1 = -(x-2)$

$\Rightarrow \boxed{y = -x + 3}$

4. The area enclosed by the curve

$$x = t^4,\ y = t - t^2$$

and the x-axis is

(a) $\frac{2}{15}$

(b) $\frac{3}{17}$

(c) $\frac{4}{15}$

(d) $\frac{1}{5}$

(e) $\frac{1}{17}$

x-intercepts: $y = 0$

$t(1-t) = 0 \Rightarrow t = 0$ or $t = 1$

$\Rightarrow \text{Area} = \int_0^1 y \cdot \left(\frac{dx}{dt}\right) dt$

$= \int_0^1 (t - t^2) \cdot 4t^3 \cdot dt$

$= \int_0^1 4(t^4 - t^5)\, dt$

$= 4\left[\frac{t^5}{5} - \frac{t^6}{6}\right]_0^1$

$= 4\left(\frac{1}{5} - \frac{1}{6}\right) = 4 \cdot \frac{1}{30} = \frac{2}{15}$

5. The graph of the polar curve $r = \sin\theta$, where $0 \le \theta < \pi$ has a horizontal tangent line when $(r, \theta) =$

(a) $\left(1, \frac{\pi}{2}\right)$

(b) $\left(\sqrt{2}, \frac{\pi}{4}\right)$

(c) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

(d) $\left(2\sqrt{2}, \frac{\pi}{4}\right)$

(e) $\left(2, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$\frac{dy}{dx} = 0 \Rightarrow \sin 2\theta = 0$$

$$\Rightarrow 2\theta = 0, \pi \Rightarrow \boxed{\theta = 0, \frac{\pi}{2}}$$

$$\theta = 0 \Rightarrow (0,0)$$

$$\theta = \frac{\pi}{2} \Rightarrow \left(1, \frac{\pi}{2}\right)$$

6. The length of the cardioid

$$r = a(1+\cos\theta), \quad a > 0$$

is

(a) $8a$

(b) a

(c) 0

(d) $4a$

(e) $\frac{a}{2}$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$$

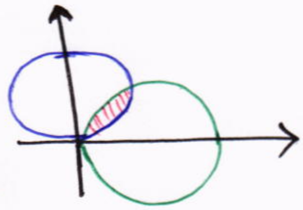
$$= 2\int_0^{\pi} \sqrt{2a^2(1+\cos\theta)}\, d\theta \qquad \boxed{a>0}$$

$$= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

7. The area of the region that lies inside both curves $r = 3\sin\theta$ and $r = 3\cos\theta$ is

$3\sin\theta = 3\cos\theta$



(a) $\frac{9\pi - 18}{8}$ $\Rightarrow \theta = \frac{\pi}{4}$

(b) $\frac{\pi + 8}{8}$

(c) $\frac{\pi}{4}$

(d) 4π

(e) $\frac{\pi - 9}{8}$

$$A = \frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} 9\cos^2\theta \, d\theta$$

By symmetry: $A = 2 \cdot \left(\frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta \, d\theta\right)$

$$A = \frac{9}{2}\int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/4} = \frac{9}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{9\pi}{8} - \frac{9}{4}$$

$$= \boxed{\frac{9\pi - 18}{8}}$$

8. The equation $x^2 + y^2 + z^2 - 2y - 4z = -5$ represents

$x^2 + (y^2 - 2y + 1) + (z^2 - 4z + 4) = 0$

(a) the point $(0, 1, 2)$

(b) a sphere with center $(2, 0, -1)$

(c) a sphere with center $(0, 1, -2)$

(d) a sphere with radius $\sqrt{5}$

(e) no graph

$x^2 + (y-1)^2 + (z-2)^2 = 0$

This eq^n represents the point $(0, 1, 2)$.