King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 208 - FINAL EXAM - Term 211

Duration: 150 minutes

MASTER

Name:______________________________ ID Number:____________________________

Section Number:____________________ Serial Number:_________________________

Instructions:

1. All types of calculators, or mobile phones are NOT allowed during the exam.
2. Use HB 2.5 pencils only.
3. Use a good eraser. DO NOT use erasers attached to the pencil.
4. The Test Code Number is already bubbled in your OMR sheet.
5. Make sure that all bubbled space on the OMR sheet is fully covered.
6. While erasing a bubbled space, make sure that you do not leave any trace of penciling.
7. **There are 15 questions, The weight of each question is 7 marks. The full mark is 105.**

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1. The eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$.
If the general solution of the system $X' = AX$ is

$$X = c_1 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_2\, e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3\, X_3,$$

then X_3 is

(a) $e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $e^{2t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(c) $e^{2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

(d) $e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(e) $e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. The solution of the initial value problem

$$X' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is

(a) $X = e^{t} \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}$

(b) $X = e^{t} \begin{bmatrix} \cos(2t) + \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}$

(c) $X = e^{-t} \begin{bmatrix} \sin(2t) - \cos(2t) \\ \sin(2t) + \sin(2t) \end{bmatrix}$

(d) $X = e^{-t} \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}$

(e) $X = e^{t} \begin{bmatrix} 2\cos(2t) - \sin(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix}$

3. The inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ is

(a) $A^{-1} = A^2 - 4A + 4I$

(b) $A^{-1} = A^2 - 4A + I$

(c) $A^{-1} = A^2 - 3A + I$

(d) $A^{-1} = -A^2 + 4A - I$

(e) $A^{-1} = -A^3 + 4A^2 - 4A + I$

4. The Jordan normal form of the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

is

(a) $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) $J = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(e) $J = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

5. Consider the system of two differential equations:

$$x'' = 2x - y$$
$$y' = x' - 2y$$

where x and y are two functions in t. Transforming this system into an equivalent system of first-order differential equations, we introduce the new functions: $x_1 = x$, $x_2 = x'$, and $x_3 = y$.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then the given system is equivalent to

(a) $X' = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} X$

(b) $X' = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} X$

(c) $X' = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} X$

(d) $X' = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} X$

(e) $X' = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} X$

6. Consider the initial value problem

$$X' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Knowing that the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ has only one eigenvalue $\lambda = 2$, and its defect is 1, the solution of the initial value problem is:

(a) $X = e^{2t} \begin{bmatrix} 2t+1 \\ 2t-1 \end{bmatrix}$

(b) $X = e^{2t} \begin{bmatrix} t+1 \\ t-1 \end{bmatrix}$

(c) $X = e^{2t} \begin{bmatrix} 1-t \\ 1 \end{bmatrix}$

(d) $X = e^{2t} \begin{bmatrix} 3t+1 \\ t-1 \end{bmatrix}$

(e) $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

7. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and consider the system $X' = AX + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Given that $e^{At} = \begin{bmatrix} e^t & 2te^t & (3t+2t^2)e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}$, the variation of parameters formula gives that **a particular solution** of the system is

(a) $X_p = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(b) $X_p = \begin{bmatrix} 4t+1 \\ -t \\ 1 \end{bmatrix}$

(c) $X_p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(d) $X_p = \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix}$

(e) $X_p = \begin{bmatrix} 2te^t \\ e^t \\ 0 \end{bmatrix}$

8. Let A be a 2×2 matrix, having two eigenvalues:

$$\lambda_1 = 1 \text{ with associated eigenvector } v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -1 \text{ with associated eigenvector } v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The exponential matrix e^{At} is

(a) $\begin{bmatrix} 2e^t - e^{-t} & e^t - e^{-t} \\ -2e^t + 2e^{-t} & -e^t + 2e^{-t} \end{bmatrix}$

(b) $\begin{bmatrix} e^t - e^{-t} & e^t - 2e^{-t} \\ -e^t + 2e^{-t} & -e^t + 4e^{-t} \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$

(d) $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$

(e) $\begin{bmatrix} e^t & e^{-t} \\ -e^t & -2e^{-t} \end{bmatrix}$

9. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, a diagonalizable matrix.

Then, a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$ (equivalently $AP = PD$) are:

(a) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(e) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

10. The general solution of the differential equation

$$y^{(3)} - 3y'' + y' + 5y = 0$$

is

(a) $y = c_1 e^{-x} + e^{2x}(c_2 \cos x + c_3 \sin x)$

(b) $y = c_1 e^{-x} + e^{x}(c_2 \cos(2x) + c_3 \sin(2x))$

(c) $y = c_1 e^{5x} + e^{-x}(c_2 \cos x + c_3 \sin x)$

(d) $y = c_1 e^{5x} + e^{2x}(c_2 \cos(2x) + c_3 \sin(2x))$

(e) $y = c_1 e^{x} + e^{4x}(c_2 \cos(2x) + c_3 \sin(2x))$

11. The vector $w = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is a linear combination of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

If $w = c_1 v_1 + c_3 v_2 + c_3 v_3$, then $c_1 + c_2 + c_3$ is equal to

(a) -1

(b) 2

(c) -4

(d) 0

(e) 3

12. Let k be a constant, and consider the system:

$$\begin{array}{ccccccc} x & - & y & & & = & 4 \\ 2x & + & 3y & + & 5z & = & 3 \\ 3x & - & y & + & kz & = & 0 \end{array}$$

The system has **a unique solution** when

(a) k is any number except 2.

(b) $k = 2$.

(c) k is only 0.

(d) k is only 3.

(e) k is any real number except -10.

13. The inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

14. Let y be the solution of the initial value problem:

$$dy - (y + xe^x)dx = 0, \qquad y(0) = 1.$$

Then, ye^{-x} equals

(a) $1 + \frac{1}{2}x^2$

(b) $1 + \frac{1}{3}x^2$

(c) $1 + \frac{1}{4}x^2$

(d) $1 + \frac{1}{5}x^2$

(e) $1 + \frac{1}{6}x^2$

15. Let $F(x, y) = 5$ be the solution of the initial value problem:

$$\frac{2x+3y}{3x+2y} + \frac{dy}{dx} = 0$$
$$y(1) = 1$$

Then, $F(-1, 1)$ equals

(a) -1.

(b) 0.

(c) 5.

(d) 4.

(e) -5.