

1. The differential equation

$$x''' + 7x'' + 9x' - 2x = \sin 2t$$

can be converted to a system of linear differential equations of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin 2t \end{bmatrix}$$

where  $A =$

(a)  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -9 & -7 \end{bmatrix}$

(correct)

(b)  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & -9 & -7 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -9 & 2 & 7 \end{bmatrix}$

(d)  $\begin{bmatrix} 2 & -9 & -7 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e)  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 9 & 7 \end{bmatrix}$

2. If  $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ , then  $e^{At} =$

(a)  $\begin{pmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{pmatrix}$

(correct)

(b)  $\begin{pmatrix} 1 + 2t & -t \\ 4t & -2t \end{pmatrix}$

(c)  $\begin{pmatrix} 1 + 2t & 1 - t \\ 1 + 4t & -2t \end{pmatrix}$

(d)  $\begin{pmatrix} 1 + 2t & 2t \\ 4t & 1 - 2t \end{pmatrix}$

(e)  $\begin{pmatrix} 2t & 1 - t \\ 4t & 1 - 2t \end{pmatrix}$

3. Let  $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}$ . If  $A^{-1} = aI + bA + cA^2$ , then  $a + b + c =$

- (a) 1
- (b) 0
- (c)  $\frac{1}{2}$
- (d)  $\frac{3}{4}$
- (e) -2

(correct)

4. Let  $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$  with  $e^{At} = \begin{pmatrix} t+1 & -t \\ t & 1-t \end{pmatrix}$ . If  $x$  is a solution of the IVP

$$x' = Ax + \begin{pmatrix} \frac{1}{t} \\ \frac{1}{t} \end{pmatrix}; x(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

where  $t > 0$ , then  $x(e) =$

- (a)  $\begin{pmatrix} 3e \\ 3e - 3 \end{pmatrix}$
- (b)  $\begin{pmatrix} 3e \\ 3e + 4 \end{pmatrix}$
- (c)  $\begin{pmatrix} 3e \\ 3e - 1 \end{pmatrix}$
- (d)  $\begin{pmatrix} 3e \\ e - 4 \end{pmatrix}$
- (e)  $\begin{pmatrix} 3e \\ e + 2 \end{pmatrix}$

(correct)

5. If the solution of the first order linear differential equation

$$y' - y = 5e^{-\frac{x}{2}}; y(0) = -2 \text{ is given by } y = \frac{Ae^x - Be^{-\frac{x}{2}}}{3}, \text{ then } 2A + B =$$

- (a) 18
- (b) 16
- (c) 24
- (d) 12
- (e) 10

(correct)

6. If the differential equation

$$(2ye^{ay} + x^2) dx + (e^y + x(2 + 6y)e^{ay}) dy = 0$$

is exact, then  $a =$

- (a) 3
- (b) 6
- (c) 2
- (d) 12
- (e) 1

(correct)

7. The form of a particular solution of

$$y'' - 2y' + y = -3e^x + 8xe^x + 10$$

is

(a)  $(Ax^2 + Bx^3)e^x + k$

(correct)

(b)  $(A + Bx)e^x + kx$

(c)  $(Ax^2 + Bx^3)e^x + kx^2$

(d)  $(Ax + Bx^2)e^x + k$

(e)  $(A + Bx)e^x + kx^2$

8. The solution of  $x' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} x$ ,  $x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  at  $t = \frac{\pi}{4}$  equals

(a)  $\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{\frac{\pi}{4}}$

(correct)

(b)  $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\frac{\pi}{4}}$

(c)  $\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{\frac{\pi}{4}}$

(d)  $\begin{pmatrix} -2 \\ 0 \end{pmatrix} e^{\frac{\pi}{4}}$

(e)  $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-\frac{\pi}{4}}$

9. The characteristic equation of  $A$  is  $(\lambda + 1)(\lambda - 5)^3 = 0$ , where we have two linearly independent eigenvectors corresponding to  $\lambda = 5$ . The Jordan normal form of  $A$  is

(a)  $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(correct)

(b)  $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(c)  $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(d)  $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(e)  $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

10. The rank of the matrix is  $\begin{pmatrix} 1 & 3 & 2 & 4 \\ -1 & -3 & 5 & 3 \\ 2 & 6 & 5 & 9 \\ 3 & 9 & 6 & m \end{pmatrix}$  is 2 if  $m =$

(a) 12

(b) 10

(c) 8

(d) 6

(e) 4

(correct)

11. The three vectors  $v_1 = (0, 3, 0)$ ,  $v_2 = (A, 2, 1)$  and  $v_3 = (B, 1, 3)$  do not form a basis for  $\mathbb{R}^3$  if

- (a)  $B = 3A$
- (b)  $B = 2A$
- (c)  $B = A$
- (d)  $B = -2A$
- (e)  $B = 5A$

(correct)

12. The general solution of  $X' = \begin{bmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} x$  can be written as

$$X = c_1 \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ b \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} e^{3t}.$$

Then  $a - 2b + 2c =$

- (a) 1
- (b) 0
- (c) 2
- (d) -1
- (e) -2

(correct)

13. (11 points) Use the method of variation of parameters to find the general solution of the differential equation

$$y'' - 2y' + y = \frac{e^x}{x^2 + 1}$$

$$y'' - 2y' + y = 0.$$

The auxiliary equation is  $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

$$y_c = c_1 e^x + c_2 x e^x. \quad (2 \text{ pts})$$

$$\text{Assume } y_p = u_1(x) e^x + u_2(x) x e^x \quad (1 \text{ pt})$$

$$W(e^x, x e^x) = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = e^{2x} \quad (1 \text{ pt})$$

$$W_1 = \begin{vmatrix} 0 & x e^x \\ \frac{e^x}{x^2+1} & e^x + x e^x \end{vmatrix} = \frac{-x e^{2x}}{x^2+1} \quad (1 \text{ pt})$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2+1} \end{vmatrix} = \frac{e^{2x}}{x^2+1} \quad (1 \text{ pt})$$

$$u_1(x) = \int \frac{W_1}{W} dx = -\int \frac{x}{x^2+1} dx = -\frac{1}{2} \ln(x^2+1) \quad (2 \text{ pts})$$

$$u_2(x) = \int \frac{W_2}{W} dx = \int \frac{1}{x^2+1} dx = \tan^{-1} x \quad (2 \text{ pts})$$

$$\Rightarrow y_p = -\frac{e^x}{2} \ln(x^2+1) + x e^x \tan^{-1} x$$

$$\text{The general soln is : } y = c_1 e^x + c_2 x e^x - \frac{e^x}{2} \ln(x^2+1) + x e^x \tan^{-1} x \quad (1 \text{ pt})$$

14. (11 points) Let  $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$ . Determine whether or not the given matrix  $A$  is diagonalizable. If it is, find a diagonalizing matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .

$$\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 10 = 0 \quad (2 \text{ pts})$$

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0 \Rightarrow \lambda = 6, -1$$

For  $\lambda = 6$ : let  $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0$  s.t.  $(A - 6I)K = 0 \Rightarrow$

$$\left[ \begin{array}{cc|c} -5 & 2 & 0 \\ 5 & -2 & 0 \end{array} \right] \xrightarrow{R_1+R_2} \left[ \begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow -5k_1 + 2k_2 = 0$$

$$\Rightarrow k_2 = \frac{5}{2}k_1, \quad K_{\lambda=6} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad (2 \text{ pts})$$

For  $\lambda = -1$ : let  $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0$  s.t.  $(A + I)K = 0 \Rightarrow$

$$\left[ \begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 5 & 0 \end{array} \right] \Rightarrow k_1 = -k_2, \quad K_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (2 \text{ pts})$$

Since  $\text{order}(A) = 2$  and it has two linearly independent eigenvectors  $\Rightarrow A$  is diagonalizable (1 pt)

$$D = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \quad (2 \text{ pts})$$

$$P = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \quad (2 \text{ pts})$$



15. (11 points) Find the general solution of  $X' = AX$  where  $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^3 = 0 \Rightarrow \lambda = 1, 1, 1 \quad (1 \text{ pt})$$

Let  $K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$  s.t.  $(A-I)K = 0 \Rightarrow$

$$\left[ \begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow 2K_2 = K_3, K_1 \text{ is free.} \quad (1 \text{ pt}) \quad (1 \text{ pt})$$

$$K = \begin{bmatrix} K_1 \\ K_2 \\ 2K_2 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} e^t \quad (1 \text{ pt})$$

To find  $X_3$  we find (nonzero) vectors  $v_1, v_2$  s.t.

$$(A-I)^2 v_2 = 0 \text{ and } (A-I)v_2 = v_1.$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 v_2 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = 0$$

$$\text{Let } v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (2 pts)} \Rightarrow v_1 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ (1 pt)}$$

$$X_3 = (v_1 t + v_2) e^t = \begin{bmatrix} -t e^t \\ 0 \\ e^t \end{bmatrix} \text{ (2 pts)}$$

The general solution is  $X = C_1 X_1 + C_2 X_2 + C_3 X_3$  (1 pt)