1. Which one of the following differential equations is **not** separable?

(a) $x\dfrac{dy}{dx} + 2y = 2xy^2$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{2xy^2-2y}{x} = \dfrac{2y(xy-1)}{x}$ not sep. (correct)

(b) $\dfrac{dy}{dx} - \sqrt{x}y = 0$ $\Rightarrow \dfrac{dy}{dx} = \sqrt{x}\cdot y$ ✓

(c) $\dfrac{dy}{dx} = 5e^{x-4y}$ $= 5e^{x}\cdot e^{-4y}$ ✓

(d) $x\dfrac{dy}{dx} - y = 2x^2y$ $\Rightarrow x\dfrac{dy}{dx} = y + 2x^2y = y(1+2x^2) \Rightarrow \dfrac{dy}{dx} = y\cdot\dfrac{1+2x^2}{x}$ ✓

(e) $\dfrac{dy}{dx} = \dfrac{x+1}{y-1}$ $= (x+1)\cdot\dfrac{1}{y-1}$ ✓

2. Find all values of a such that $y = ax - \ln x$ is a solution of the differential equation

$$x^2y'' + xy' - y = \ln x.$$

$y' = a - \dfrac{1}{x}$

$y'' = \dfrac{1}{x^2}$

Sub. in the ODE:

(a) $a \in (-\infty, \infty)$ (correct)

(b) $a \in (0, \infty)$

(c) $a \in (-\infty, 0)$

(d) $a \in [1, \infty)$

(e) $a \in [-1, \infty)$

$LHS = x^2y'' + xy' - y = x^2\left(\dfrac{1}{x^2}\right) + x\left(a - \dfrac{1}{x}\right) - (ax - \ln x)$

$= 1 + ax - 1 - ax + \ln x$

$= \ln x$ for all $a \in (-\infty, \infty)$

$= RHS$

3. If the system

$$\begin{aligned} 4x + 3y &= 5 \\ 8x + ky &= 10 \end{aligned}$$

has infinitely many solutions, then $k =$

(a) 6 (correct)

(b) 3

(c) 4

(d) -3

(e) -4

$$\left[\begin{array}{cc|c} 4 & 3 & 5 \\ 8 & k & 10 \end{array}\right] \xrightarrow{R_2 \to -2R_1 + R_2} \left[\begin{array}{cc|c} 4 & 3 & 5 \\ 0 & k-6 & 0 \end{array}\right]$$

$\hookrightarrow$ if $k-6=0$, i.e., $k=6$, then there are infinitely many sol.

[y is a free variable].

4. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

If

$$A^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{bmatrix},$$

then $a+b+c+d =$

(a) -2 (correct)

(b) $-\frac{3}{4}$

(c) $-\frac{1}{2}$

(d) $\frac{1}{4}$

(e) $\frac{5}{4}$

$$\begin{array}{c} \begin{array}{cc} A & I \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\xrightarrow{R_2 \to \frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -2R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 5/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -\frac{1}{2}R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 5/4 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{5}{4}R_3 + R_2} \begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1/4 & -5/4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cc} I & A^{-1} \end{array} \end{array}$$

$a = -\frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{4}$, $d = -\frac{5}{4}$

$a+b+c+d = -2$

5. If the differential equation

$$\underbrace{(Ax^2y + (A - B)y\sin(2x))}_{M}\,dx + \underbrace{(x^3 + B\sin^2 x)}_{N}\,dy = 0$$

is **exact**, then

$M_y = Ax^2 + (A-B)\sin(2x)$

$N_x = 3x^2 + B \cdot 2\sin x \cos x$

$= 3x^2 + B\sin(2x)$

$M_y = N_x \Rightarrow A = 3 \;\&\; A - B = B$

$\Rightarrow A = 3 \;\&\; 3 - B = B$

$\Rightarrow A = 3 \;\&\; B = \frac{3}{2}$

(a) $A = 3$, $B = 3/2$ (correct)

(b) $A = 2$, $B = 2/3$

(c) $A = 3$, $B = -3/2$

(d) $A = 1$, $B = -1$

(e) $A = -1$, $B = -2$

6. Find the **determinant** of the matrix

$$\begin{array}{c} \\ \rightarrow \\ \\ \\ \\ \end{array} A = \begin{array}{cccc} + & - & & \\ \end{array}$$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

$$\det(A) = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= 2\left[2(3) - (2) + 0\right] - \left[1(3) - 1(0) + 0\right]$

$= 2(4) - 3 = 8 - 3 = 5$

(a) 5 (correct)

(b) -4

(c) 0

(d) 8

(e) -3

7. A one-parameter family of solutions of the differential equation

$$\frac{dy}{dx} = 2xy + 2x + y + 1 = 2x(y+1) + (y+1) = (y+1)(2x+1) \quad \text{sep.}$$

is

$$\Rightarrow \frac{1}{y+1}\,dy = (2x+1)\,dx$$

$$\Rightarrow \ln|y+1| = x^2 + x + c_1$$

$$\Rightarrow |y+1| = e^{x^2+x+c_1}$$

$$\Rightarrow y+1 = \underbrace{\pm e^{c_1}}_{c} \cdot e^{x^2+x}$$

$$\Rightarrow y = -1 + c\,e^{x^2+x}.$$

(a) $y = -1 + ce^{x^2+x}$ (correct)

(b) $y = 1 + ce^{x^2+x}$

(c) $y = -2 + ce^{2x^2+x}$

(d) $y = -1 + ce^{x^2-2x}$

(e) $y = 1 + ce^{x^2-x}$

8. If a ~~particular~~ particle moving in a straight line has ~~a~~ acceleration

$$a(t) = \frac{3}{(t+1)^3},$$

an initial position $x(0) = 0$, and an initial velocity $v(0) = 0$, then $x(2) =$ (the position of the particle at $t = 2$)

$$a(t) = 3(t+1)^{-3}$$

$$v(t) = \int 3(t+1)^{-3}\,dt = -\tfrac{3}{2}(t+1)^{-2} + C$$

$$v(0) = 0 \Rightarrow 0 = -\tfrac{3}{2} + C \Rightarrow C = \tfrac{3}{2}$$

$$v(t) = -\tfrac{3}{2}(t+1)^{-2} + \tfrac{3}{2}$$

$$\Rightarrow x(t) = \int -\tfrac{3}{2}(t+1)^{-2} + \tfrac{3}{2}\,dt$$

$$= \tfrac{3}{2}(t+1)^{-1} + \tfrac{3}{2}t + D$$

$$x(0) = 0 \Rightarrow 0 = \tfrac{3}{2} + 0 + D \Rightarrow D = -\tfrac{3}{2}$$

$$\text{So } x(t) = \tfrac{3}{2}(t+1)^{-1} + \tfrac{3}{2}t - \tfrac{3}{2}$$

$$x(2) = \tfrac{3}{2}(3)^{-1} + \tfrac{3}{2}(2) - \tfrac{3}{2}$$

$$= \tfrac{1}{2} + 3 - \tfrac{3}{2} = 3 - 1 = 2$$

(a) 2 (correct)

(b) $\frac{3}{2}$

(c) 4

(d) $\frac{10}{3}$

(e) $\frac{9}{4}$