**KFUPM/ Department of Mathematics/T221/MATH 210/ Final Exam/**

**Name:** **ID #:** **Serial #:**

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**1.** [12pts] (a) Give an example of a denumerable subset $X$ of $\mathbb{R}$ and an uncountable subset $Y$ of $\mathbb{R}$ such that $X \cap Y$ is nonempty and finite.

(b) Let $A$ and $B$ be sets such that $A \subseteq B$. Prove that if $A$ is uncountable, then $B$ is uncountable.

(c) Using Schröder-Bernstein Theorem or otherwise, prove that $(0,1) \approx [0,1)$.

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**2.** [12pts] (a) Let $p$ and $q$ be distinct primes.

(i) Prove that $\gcd(p, p+q) = 1$.

(ii) Prove that if $p$ and $q$ are divisors of an integer $a$, then $pq|a$.

(b) State the fundamental theorem of arithmetic, and prove that if $a$ and $b$ are integers that are not coprime, then there exists a prime $p$ dividing both $a$ and $b$.

(c) Let $m, n$ be integers such that $m \geq 2$, $2n \equiv 3 \,(\operatorname{mod} m)$ and $2n^2 \equiv 7 \,(\operatorname{mod} m)$. Find $m$.

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**3.** [12pts] (a) Let $\mathbb{Q}^*$ denote the multiplicative group of nonzero rational numbers and let $G$ be the set of all rational numbers of the form $2^k$, where $k \in \mathbb{Z}$. Prove that $G$ is a subgroup of $\mathbb{Q}^*$.

(b) Let $H$ be an Abelian group and let $f : H \longrightarrow H$ be the function given by $f(a) = a^{-1}$ for each $a$ in $H$. Prove that $f$ is an isomorphism.

(c) A group $G$ of order 20 has a subgroup $A$ of order $\alpha$ and a subgroup $B$ of order $\beta$. If $10 \leq 2\alpha < \beta$, find all possible values of $\alpha$ and $\beta$.

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**4.** [12pts] (a) Let $R$ be the relation on $\mathbb{Q}$ defined by $aRb$ iff $3a - 3b \in \mathbb{Z}$. Prove that $R$ is an equivalence relation and determine the equivalence class $[1/3]$.

(c) Let $T$ be the relation on $\mathbb{R}^*$ defined by $aTb$ iff $\dfrac{a}{b} = 2^k$ for some nonnegative integer $k$. Is $(\mathbb{R}^*, T)$ a poset? is it well-ordered? Justify your answers. [$\mathbb{R}^*$ is the set of all nonzero real numbers.]

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**5.** [12pts] Mark each of the following statements as True or False and justify your choices.

(a) For all positive integers $a, b, c$, if $a|bc$ then $a|b$ or $a|c$.

(b) The function $g : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ given by $g(x) = (x, 2x)$ for all $x \in \mathbb{R}$ is a bijection.

(c) The subset $\{[0], [3], [6], [9], [12], [15], [18]\}$ of $\mathbb{Z}_{20}$ is a cyclic subgroup of $\mathbb{Z}_{20}$ (under addition).