King Fahd University of Petroleum & Minerals
Department of Mathematics

Math 225 Introduction to Linear Algebra (Term 221)
Midterm Exam (Duration = 100 minutes)

Exercise 1

Given the linear systems

$$\text{(i)}\quad \begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 + 3x_2 &= 0 \end{aligned} \quad \text{and} \quad \text{(ii)}\quad \begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ -x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_2 &= -2 \end{aligned}$$

Solve both systems by computing the reduced row echelon form of an augmented matrix $(A|B)$

Exercise 2

Let $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 1 & 1 & 2 \end{pmatrix}$ and set $A^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$.

Compute $x_{22}, x_{21}, x_{13}$by using ***Cramer's rule***.

Exercise 3

Consider the linear operator on P_3 defined by $L(p) = p - p'$.

(a) Find the kernel of L
(b) Find the range of L
(c) Is L one-to-one and why?
(d) Is L onto and why?

Exercise 4

Let S be the subspace of P_3 consisting of all polynomials $p(x)$ such that $p(2) = 0$, and let T be the subspace of all polynomials $q(x)$ such that $q(-1) = 0$. Find bases for:

(a) S (b) T (c) $S \cap T$ (d) $S + T$

Exercise 5

The linear transformation defined by $L(p) = p'(x) + p(1)$ maps P_3 into P_2.

(a) Find the matrix representation of L with respect to the ordered bases $\{x^2, x, 1\}$ and $\{2, 1 + x\}$.
(b) Use this matrix to find the coordinates of $L(x^2 - 1)$ with respect to the ordered basis $\{2, 1 + x\}$.

Exercise 6

Let A and B be two similar matrices and let c be a real number.

(a) Show that $\det(A) = \det(B)$.

(b) Show that $\det(A - cI) = \det(B - cI)$.

Exercise 7

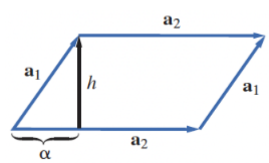
Let $x = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \end{bmatrix}$.

(a) Determine the angle θ between x and y.

(b) Determine the distance between x and y.

Exercise 8

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$. The vectors a_1 and a_2 of A are used to form a parallelogram with altitude h, and α as shown in the figure:



Determine α and use a relation between α and h to show that $|\det(A)|$ is equal to the area of the parallelogram.

Exercise 9

The vectors $x_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $x_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ form an orthonormal set in $\mathbb{R}^4$.

Extend this set to an orthonormal basis for $\mathbb{R}^4$.

Exercise 10

Let K be an $n \times n$ matrix of the form

$$K = \begin{pmatrix} 1 & -c & -c & \cdots & -c & -c \\ 0 & s & -sc & \cdots & -sc & -sc \\ 0 & 0 & s^2 & \cdots & -s^2c & -s^2c \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & s^{n-2} & -s^{n-2}c \\ 0 & 0 & 0 & \cdots & 0 & s^{n-1} \end{pmatrix}$$

where $c = \cos\theta$ and $s = \sin\theta$, for some angle $\theta \in [-\pi, \pi]$.

(a) If k_j denotes the jth column of K, find $\|k_j\|^2$

(b) If $n = 100$, find $\|K\|_F$