KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics and Statistics

Math 302 First Major Exam

Semester (212)

Feb 23, 2022 (100 Minutes)

Name: ....KEY..........

I.D: .... Section: .... Serial #: ....

Instructions

1. No electronic device (such as calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/15
2	/15
3	/15
4	/15
5	/15
Total	/75

Question 1 (5+5+5 points)

Consider the set $E = \{(a, b, c) \in \mathbb{R}^3 \mid b - 2c = 0\}$

a. Show that E is a subspace of $\mathbb{R}^3$.

* Closed under addition:

If $(a_1, b_1, c_1), (a_2, b_2, c_2) \in E$

$\Rightarrow (a_1+a_2, b_1+b_2, c_1+c_2) \in E$ because

$b_1 - 2c_1 = 0$ and $b_2 - 2c_2 = 0 \Rightarrow (b_1+b_2) - 2(c_1+c_2) = 0$

* closed under scalar Multiplication:

$k(a_1, b_1, c_1) = (ka_1, kb_1, kc_1) \in E$ because

$b_1 - 2c_1 = 0 \Rightarrow k(b_1 - 2c_1) = 0$

b. Find a basis and dimension of E.

Let $(a,b,c) \in E \Rightarrow$

$$(a,b,c) = (a, 2c, c)$$
$$= a(1,0,0) + c(0,2,1)$$

Thus the basis $B \{(1,0,0), (0,2,1)\}$

Dim = 2

c. Write $v = (1,6,3) \in E$ as a linear combination of vectors in that basis.

$v = (1,6,3) = \alpha(1,0,0) + \beta(0,2,1)$

$\Rightarrow \alpha = 1$

$2\beta = 6 \Rightarrow \beta = 3$

Question 2 (3+3+3+3+3 points)

Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 5 & k \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

a. Find k.

$R_1 \cdot C_3 = 0 \Rightarrow 2k + 0 + 6 = 0$

$\Rightarrow k = -3$

b. Find $(\mathbf{A}^T)^{-1}$.

$$(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} -2 & -8 & 5 \\ 5 & 17 & -10 \\ -3 & -10 & 6 \end{pmatrix}$$

c. What is the rank of $\mathbf{A}$ and the rank of $[\mathbf{A}|\mathbf{B}]$? Explain.

Rank A = Rank $[A|B] = 3$

because A^{-1} exists and the solution exists for the system $AX = B$

d. Solve the system $\mathbf{AX} = \mathbf{B}$.

$$X = A^{-1}B = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$$

e. What type of solutions do the homogenous system $\mathbf{AX} = \mathbf{O}$ have? where $\mathbf{O}$ is the zero 3×1 matrix.

$AX = 0$

$X = A^{-1} 0 = 0$

So, we have only the trivial solution

Question 3 (9+6 points)

Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$.

Eigenvalues of **A** are $\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 5$ and the corresponding eigenvectors are $X_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, respectively.

a. Find X_2.

$$(A-3I) = \begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad 3$$

$$\to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad 3$$

$$X_2 = \begin{pmatrix} -2t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad 3$$

For $t = -1$

b. Find the eigenvalues and eigenvectors for A^{-1}.

$\lambda_1' = 1$ and $X_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\lambda_2' = \frac{1}{3}$ and $X_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ 1 each

$\lambda_3' = \frac{1}{5}$ and $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Question 4 (5+10 points)

Given that the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

a. Find all eigenvalues of **A** and their corresponding eigenvectors.

$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2+1=0 \Rightarrow \lambda_1 = i,\ \lambda_2 = -i$ 1 1

$(A - iI) = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \to \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} it \\ t \end{pmatrix} = t\begin{pmatrix} i \\ 1 \end{pmatrix}$ 2

and $X_2 = \overline{X_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ 1

b. Show that $\mathbf{A}^{10}$ is diagonalizable and find its eigenvalues.

A has two eigenvalues $\Rightarrow$ diagonilizable

$P^{-1} A P = D$

$P^{-1} A^n P = D^n$, there exist P such that $P^{-1} A^n P = D^n$

$\Rightarrow A^n$ is diagonlizable, indeed A^{10} is diag. 5

Eigenvalues of A^{10} are the same for D^{10}

$D^{10} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{10} = \begin{bmatrix} i^{10} & 0 \\ 0 & (-i)^{10} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ 3

so, we have $\lambda_1 = \lambda_2 = -1$

1 1

Question 5 (5+5+5 points)

Consider a symmetric 3×3 matrix **A** with eigenvalues

$\lambda_1 = 3,\ \lambda_2 = 6,\ \lambda_3 = 9$ and the corresponding eigenvectors are

$X_1 = \begin{bmatrix} k \\ 2 \\ 1 \end{bmatrix},\ X_2 = \begin{bmatrix} k \\ -1 \\ -2 \end{bmatrix},\ X_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, respectively.

a. Find k.

$X_1 \cdot X_2 = k^2 - 2 - 2 = 0 \Rightarrow k^2 = 4 \Rightarrow k = \mp 2$

we exclude $k = -2$ to garantee $X_2 \cdot X_3 = 0$

Thus, $k = 2$

5

b. Find an orthogonal matrix **P** that diagonalizes **A** and the diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

$$P = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \end{pmatrix}$$

3

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

2

b. Find $\mathbf{P}^{-1}$ that verifies the equation $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

$$P^{-1} = P^T = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \end{pmatrix}$$

5

because P is an orthogonal matrix.