KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics and Statistics

Math 302 Second Major Exam

Semester (212)

March 23, 2022 (100 min)

Name: ....KEY..........

I.D: .... Section: .... Serial #: ....

Instructions

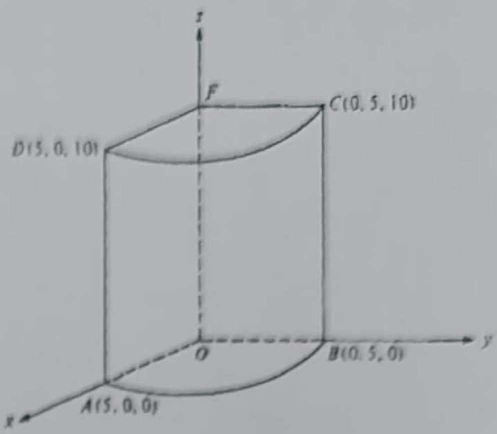
1. No electronic device (such as calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/15
2	/15
3	/15
4	/15
5	/15
Total	/75

Question 1 (5+5+5 points)

Consider the shape in the adjacent figure. Calculate:

a. the length of AB.



$$AB = \int_0^{\pi/2} \rho \, d\phi = \int_0^{\pi/2} 5 \, d\phi$$ 3

$$= \frac{5\pi}{2}$$ 2

b. the surface area of $ABCD$.

$$S = \int_S dS = \int_0^{10}\int_0^{\pi/2} \rho \, d\phi \, dz = 5(10)\left(\frac{\pi}{2}\right) = 25\pi$$ 3 2

c. the volume of $ABDCFO$.

$$V = \iiint \rho \, d\rho \, d\phi \, dz$$

$$= \int_0^{10}\int_0^{\pi/2}\int_0^{5} \rho \, d\rho \, d\phi \, dz$$

$$= \left(\int_0^5 \rho \, d\rho\right)\left(\int_0^{\pi/2} d\phi\right)\left(\int_0^{10} dz\right)$$ 3

$$= \frac{25}{2} \cdot \frac{\pi}{2} \cdot 10 = \frac{125}{2}\pi$$ 2

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Question 2 (5+10 points)

a. A point charge of 5 nC is located at the origin. If $V = 2\text{V}$ at the point $P(0,6,-8)$, find the electric potential at $B(1,5,7)$.

$$V_B - V_P = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_P}\right] = 45\left[\frac{1}{\sqrt{75}} - \frac{1}{10}\right] \quad 3$$

$$V_B = 2 + 45\left[\frac{1}{5\sqrt{3}} - \frac{1}{10}\right] = 2 + 45\left[\frac{2-\sqrt{3}}{10\sqrt{3}}\right]$$

$$= 2 + \frac{6\sqrt{3}-9}{2} \quad 2$$

$$= \frac{6\sqrt{3}-5}{2}$$

b. Given the electrostatic field

$$E = \sin^2\theta\cos\phi\, a_r + \sin 2\theta\cos\phi\, a_\theta - \sin\theta\,\cancel{\cos}\overset{\sin}{}\phi\, a_\phi.$$

Calculate the work done in moving a $10\mu C$ charge from the point $(1, 30^\circ, 45^\circ)$ to the point $(2, 30^\circ, 90^\circ)$.

$$E = -\nabla V$$

$$\Rightarrow V = -r\sin^2\theta\cos\phi$$

$$W = -Q\int_{(1,30^\circ,45^\circ)}^{(2,30^\circ,90^\circ)} E\cdot dl = QV_{AB}$$

$$= -Q\left[V(2,30^\circ,90^\circ) - V(1,30^\circ,45^\circ)\right]$$

$$= -10\mu$$

Question 3 (5+10 points)

Consider $\mathbf{D} = \rho^2 \cos^2 \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_\phi$.

a. Find div **D**.

$$\nabla \cdot D = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho(\rho^2 \cos^2\phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (z \sin\phi)$$

$$= \frac{1}{\rho}(3\rho \cos^2\phi + \frac{1}{\rho}(z \cos\phi)$$

$$= 3\rho \cos^2\phi + \frac{z}{\rho} \cos\phi$$

b. Use the divergence theorem to calculate the flux of **D** out the closed surface of the cylinder $0 \le z \le 1, \ \rho = 4$.

$$\text{Flux} = \oint D \cdot dS = \int_v \nabla \cdot D \, dv$$

$$= \int_0^1 \int_0^{2\pi} \int_0^4 (3\rho \cos^2\phi + \frac{z}{\rho} \cos\phi) \rho \, d\rho \, d\phi \, dz$$

5

$$= 64\pi$$

5

Question 4 (5+5+5 points)

Consider $\mathbf{B} = (y + z\cos xz)\mathbf{a}_x + x\mathbf{a}_y + x\cos xz\ \mathbf{a}_z$.

a. Show that **B** is conservative.

$$\text{curl } B = \nabla \times B = \begin{vmatrix} a_x & a_y & a_z \\ \partial_x & \partial_y & \partial_z \\ y + z\cos xz & x & x\cos xz \end{vmatrix}$$

$$= (0-0)a_x - (\cos xz - xz\sin xz - \cos xz + zx\sin xz)\vec{a_y} + (1-1)a_z$$
$$= \vec{0}$$

b. Find the potential function ϕ of **B**.

$$B = \nabla\phi$$
$$\phi_x = y + z\cos xz$$
$$\phi = yx + \sin xz + F(y,z)$$
$$\phi_y = x + \frac{\partial F}{\partial y} = x \Rightarrow \frac{\partial F}{\partial y} = 0 \Rightarrow F(y,z) = G(z)$$
$$\phi_z = x\cos xz + G'(z) = x\cos xz \Rightarrow G'(z) = 0 \Rightarrow G(z) = C$$

$$\Rightarrow \phi = xy + \sin xz$$

c. Evaluate $\int_{(1,0,\pi)}^{(2,1,\pi)} \boldsymbol{B}.d\boldsymbol{l}$.

$$I = \phi(2,1,\pi) - \phi(1,0,\pi) = 2 - 0 = 2$$

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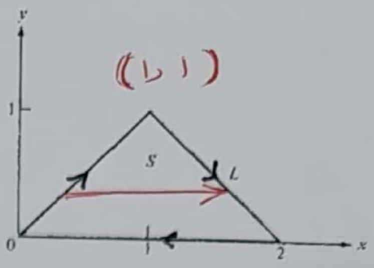
Question 5 (15 points)

Use Stockes's theorem to evaluate $\oint_L \mathbf{F}.\,d\mathbf{l}$, where $\mathbf{F} = xy^2\boldsymbol{a}_x - y\boldsymbol{a}_y$ and L is shown in the adjecent figure.



$$\oint_L F.dI = \int_S \nabla\times F \, dS$$

$$\nabla\times F = \begin{vmatrix} a_x & a_y & a_z \\ \partial x & \partial y & \partial z \\ xy^2 & -y & 0 \end{vmatrix} = -2yx\,\vec{a}_z \quad 3$$

$dS = dx\,dy\,(-a_z)$ 3 (right hand rule)

$$\oint_L F.dI = \int_0^1 \int_y^{2-y} (2yx)\,dx\,dy \quad 6$$

$$= \frac{2}{3} \quad 3$$

OR $\int_0^1 \int_0^x (2yx)\,dy\,dx + \int_1^2 \int_0^{2-x} (2yx)\,dy\,dx$

$= 2/3$

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