King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 323 (Term 212)
Midterm Exam (Duration = 120 minutes)

Exercise 1 (6 points). Answer the following questions and justify your answers.

- **(a)** Is $\left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^{\star}$, under multiplication, a group?
- **(b)** Is the group $(\mathbb{Q}, +)$ cyclic?
- **(c)** Is there an integer $n \geq 3$ such that the symmetric group S_n is abelian?

Exercise 2 (6 points). Let G be a group of order $n = pqr$, where $p \lneqq q \lneqq r$ are prime numbers. Let H be a *proper* subgroup of G such that $\exists\, a, b \in H \setminus \{1\}$ with $|a| \neq |b|$. Find, all possible values for $|H|$.

Exercise 3 (6 points). Let $G := \frac{\mathbb{Z}}{n\mathbb{Z}}$, where $n = pqr$ and $2 \lneqq p \lneqq q \lneqq r$ are prime numbers. Let $H = \langle \overline{m} \rangle$ be a subroup of G of order pq such that $1 < m \lneqq n - r$. Find the largest possible value for m.

Exercise 4 (6 points). Let S_{12} denote the symmetric group of degree 12 and let $\sigma \in S_{12}$ with $\sigma = \alpha_1\alpha_2 \cdots \alpha_k$ where the α_i's are disjoint r_i-cycles such that $r_i \geq 4$, for each i, and $r_1 + r_2 + \cdots + r_k = 12$. Determine all possible values for the order of σ and, for each case, state whether σ is even or odd.

Exercise 5 (6 points). Draw the lattice of all subgroups of $\frac{\mathbb{Z}}{66\mathbb{Z}}$ with all possible inclusions between subgroups and, for each subgroup, give its generator and order.

Exercise 6 (10 points). Let $G = \frac{\mathbb{Z}}{p^2\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is a prime number. Find the number of elements of G of order p.

Exercise 7 (10 points). Let $G = U(66)$ be the multiplication group modulo 66.

- **(a)** Determine the isomorphism class of G.
- **(b)** Express G as an internal direct product of cyclic groups.

[In G, we have: $5^2, 5^3, 5^4, 5^5, 5^6, 5^7, 5^8, 5^9, 5^{10} = 25, 59, 31, 23, 49, 47, 37, 53, 1$, respectively ; $17^2, 17^3, 17^4, 17^5, 17^6, 17^7, 17^8, 17^9, 17^{10} = 25, 29, 31, 65, 49, 41, 37, 35, 1$, respectively ; $23^2 = 43^2 = 65^2 = 1$.]

Exercise 8 (10 points). Show that $\mathbb{Q}$, under addition, has no *proper* subgroup of finite index

---

Exercise 9 (20 points). Let G be a *non-trivial* finite group. Consider the *equivalence relation R* defined on G by xRy if $y = gxg^{-1}$ for some $g \in G$. For each $x \in G$, let cl(x) denote the equivalence class of x and let C(x) denote the centralizer of x. That is, $\mathrm{cl(x)} := \{\mathrm{gxg^{-1}} \mid \mathrm{g} \in \mathrm{G}\}$ and $\mathrm{C(x)} := \{\mathrm{g} \in \mathrm{G} \mid \mathrm{gx} = \mathrm{xg}\}$.

- **(a)** Prove that $|\mathrm{cl(x)}| = |\mathrm{G} : \mathrm{C(x)}|$ for each $x \in G$.
- **(b)** Show that $|G| = \sum_{x \in G} |G : \mathrm{C(x)}|$, where the sum runs over *one* representative x of cl(x).
- **(c)** Show that $x \in Z(G)$ if and only if $|G : \mathrm{C(x)}| = 1$.
- **(d)** Assume $|G| = p^n$, where p is a prime number and $n \geq 1$. Prove that $Z(G)$ is NOT trivial.
- **(e)** Deduce from (d) that if $|G| = p^2$, then G is abelian.