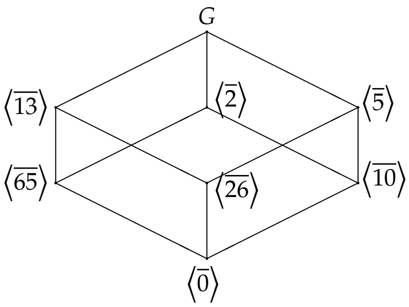
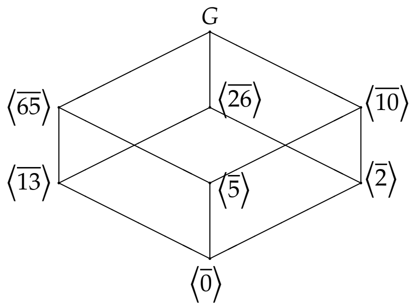
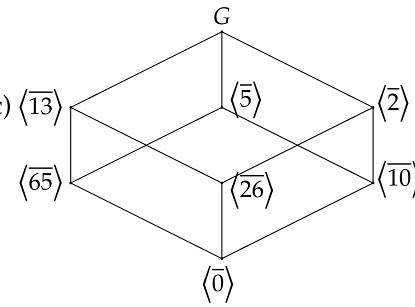
King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

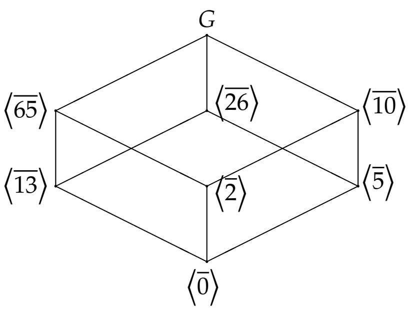
Final Exam – Math 323 (Term 212)
(Duration = 3 hours | Number of Questions = 24) (**CODE** 001)

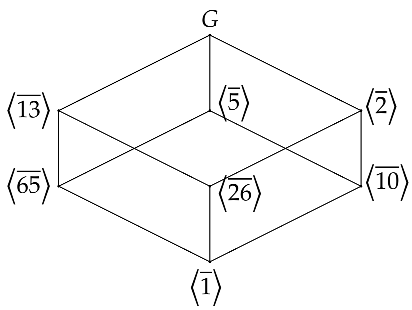
Exercise 1. The *lattice* of all subgroups of $\dfrac{\mathbb{Z}}{130\mathbb{Z}}$ is.

(a)


(b)


(c)


(d)


(e)


Exercise 2. In the group $\dfrac{\mathbb{Z}}{69\mathbb{Z}}$, let H_1 and H_2 be two distinct non-trivial proper subgroups. Then, $|H_1 \cup H_2| =$

a) 26
b) 25
c) 69
d) 45
e) 46

Exercise 3. Let G be a group of order 4. Which one of the following statements is WRONG?

a) There must be at least one subgroup of G of order 2
b) G may or may not be cyclic
c) There must be an element of G of order 4
d) G must be abelian
e) Every proper subgroup of G is cyclic

Exercise 4. In the group $\frac{\mathbb{Z}}{70\mathbb{Z}}$, let H be a subgroup of order 10. If $H = \langle \overline{n} \rangle$ such that $n \leq 60$, then the largest possible value for n is equal to:

a) 7

b) 10

c) 21

d) 49

e) 56

Exercise 5. Let S_{10} denote the symmetric group of degree 10 and let $\sigma \in S_{10}$ with $\sigma = \alpha_1\alpha_2\cdots\alpha_k$ where the α_i's are disjoint r_i-cycles such that $r_i \geq 3$, for each i, and $r_1 + r_2 + \cdots + r_k = 10$. If σ is odd, then the largest possible order for σ is equal to:

a) 6

b) 8

c) 12

d) 21

e) 30

Exercise 6. Let G be a non-abelian group of order pq, where $p < q$ are prime numbers. Then:

a) $Z(G)$ is trivial

b) $Z(G)$ has an element of order q

c) $|Z(G)| = p$

d) $\frac{G}{Z(G)}$ is cyclic

e) $\left|\frac{G}{Z(G)}\right| = 1$

Exercise 7. Let G be a group of order $n = 2pq$, where $2 \lneqq p \lneqq q$ are prime numbers and let H be a non-cyclic subgroup of G such that $|H|$ is odd. If x is an element of G of order $2p$, then

a) $|xH| = 2$

b) $|xH| = p$

c) $|xH| = q$

d) $xH = Hx$

e) $xH = H$

Exercise 8. Let $G = U(66)$ be the group under multiplication modulo 66. In G, we have:
$5^2, 5^3, 5^4, 5^5, 5^6, 5^7, 5^8, 5^9, 5^{10} = 25, 59, 31, 23, 49, 47, 37, 53, 1$, respectively ;
$17^2, 17^3, 17^4, 17^5, 17^6, 17^7, 17^8, 17^9, 17^{10} = 25, 29, 31, 65, 49, 41, 37, 35, 1$, respectively ;
$23^2 = 43^2 = 65^2 = 1$. Then:

a) $G \cong \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$ and $G = \langle 43 \rangle \oplus \langle 5 \rangle$

b) $G \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{11\mathbb{Z}}$ and $G = \langle 43 \rangle \oplus \langle 5 \rangle$

c) $G \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{10\mathbb{Z}}$ and $G = \langle 23 \rangle \oplus \langle 5 \rangle$

d) $G \cong \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$ and $G = \langle 65 \rangle \oplus \langle 5 \rangle$

e) $G \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{10\mathbb{Z}}$ and $G = \langle 23 \rangle \oplus \langle 17 \rangle$

Exercise 9. Let G be a non-abelian group of order p^3, where p is prime.

a) $Z(G) \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$

b) $Z(G)$ is trivial

c) $Z(G) \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$

d) $Z(G) \cong \frac{\mathbb{Z}}{p^2\mathbb{Z}}$

e) None of these statements are true

Exercise 10. Which one of the following statements is WRONG?

a) $\frac{\mathbb{Z}}{3\mathbb{Z}}[i]$ is a field

b) $\frac{\mathbb{Z}}{17\mathbb{Z}}[i]$ is NOT a field

c) $\frac{\mathbb{Z}}{11\mathbb{Z}}[i]$ is a finite ring

d) $\frac{\mathbb{Z}}{13\mathbb{Z}}[i]$ is a field

e) $\frac{\mathbb{Z}}{5\mathbb{Z}}[i]$ is NOT an integral domain

Exercise 11. Consider the ring $R := \mathbb{Z} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is a prime number. Let Q be a prime ideal of R and let $(n, \overline{m}) \in Q$ such that $n \geq 1$ and $1 \leq m \leq p-1$. If n_o = *Smallest possible value for* n and m_o = *Largest possible value for* m, then $n_o + m_o$ is equal to:

a) $p-2$

b) $p-1$

c) p

d) $p+1$

e) $p+2$

Exercise 12. Which one of the following statements is CORRECT?

a) In the ring $\frac{\mathbb{Z}}{5\mathbb{Z}}[i]$, the principal ideal $(1+i)\frac{\mathbb{Z}}{5\mathbb{Z}}[i]$ is prime

b) $\frac{\mathbb{Z}[i]}{i\mathbb{Z}[i]}$ is isomorphic to $\mathbb{Z}$

c) The ring of 2×2 matrices over $\frac{\mathbb{Z}}{2\mathbb{Z}}$ is a division ring

d) The ideal $0 \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ is not maximal in the ring $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$

e) In the ring $\mathbb{Z}[i]$, the principal ideal $(1-i)\mathbb{Z}[i]$ is maximal

Exercise 13. Consider the commutative ring $R := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ and the two ideals of R $I := \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \mid a \in \mathbb{Z} \right\}$ and $J := \left\{ \begin{pmatrix} a & -a \\ -a & a \end{pmatrix} \mid a \in \mathbb{Z} \right\}$. Consider the mapping $\phi : R \longrightarrow \mathbb{Z}\,;\, \begin{pmatrix} a & b \\ b & a \end{pmatrix} \longmapsto a + b$. Which one of the following statements is WRONG?

a) ϕ is a ring homomorphism

b) I is a prime ideal of R

c) $\dfrac{R}{J} \cong \mathbb{Z}$

d) $\mathrm{Ker}(\phi) = \mathrm{I}$

e) $\phi(R) = \mathbb{Z}$

Exercise 14. Let n be a positive integer with decimal representation $ababc$. If n is divisible by 7 and $3a + b = 12$, then c is equal to:

a) 1

b) 2

c) 3

d) 4

e) 5

Exercise 15. In $\dfrac{\mathbb{Z}}{13\mathbb{Z}}[X]$, let $\bar{r}$ be the remainder of the division of X^n by $X + 5$, with $0 \leq r \leq 12$. If $n = 43$, then r is equal to:

a) 1

b) 2

c) 3

d) 4

e) 5

Exercise 16. The equation $x^{70} = 1$ (*modulo* 61) has

a) 6 distinct solutions

b) 10 distinct solutions

c) 11 distinct solutions

d) 14 distinct solutions

e) 15 distinct solutions

Exercise 17. Let F be a finite field of characteristic p.

a) $x^p = 1$, $\forall\, x \in F^\star$

b) $F \cong \dfrac{\mathbb{Z}}{p\mathbb{Z}}$

c) $p \leq |F|$

d) $x^p = x$, $\forall\, x \in F$

e) $F \times F$ is a field of characteristic p

Exercise 18. Which one of the following polynomials is NOT irreducible in $\mathbb{Q}[X]$?

a) $2X^4 + 4X^2 + 2$

b) $X^3 + 2X^2 + X - 1$

c) $2X^3 + X^2 + 3X + 2$

d) $2X^3 + 4X^2 + 6X + 8$

e) $X^4 + 4X^2 + 6$

Exercise 19. Which one of the following factor rings is a field?

a) $\frac{\mathbb{Z}_3[X]}{(X^4+1)}$

b) $\frac{\mathbb{Z}_3[X]}{(X^4-1)}$

c) $\frac{\mathbb{Z}_3[X]}{(X^4+2X+2)}$

d) $\frac{\mathbb{Z}_3[X]}{(X^4+2X+1)}$

e) No one of these factor rings is a field

Exercise 20. The polynomial $f := X^2+1$ is NOT irreducible in

a) $\mathbb{Z}_{11}[X]$

b) $\mathbb{Z}_{19}[X]$

c) $\mathbb{Z}_{23}[X]$

d) $\mathbb{Z}_{31}[X]$

e) f is irreducible in all these polynomial rings

Exercise 21. The polynomial X^4-X^2+1 is NOT irreducible over

a) $\mathbb{Q}$

b) $\mathbb{Q}[\sqrt{2}]$

c) $\mathbb{Q}[\sqrt{3}]$

d) $\mathbb{Q}[i]$

e) f is irreducible over all these fields

Exercise 22. The ring $\frac{\mathbb{Q}[X]}{(X^3-X^2-2X+2)}$ is isomorphic to

a) $\mathbb{Q}\times\mathbb{Q}[\sqrt{2}]$

b) $\mathbb{Q}\times\mathbb{Q}[i]$

c) $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$

d) $\mathbb{Q}\times\mathbb{C}$

e) $\mathbb{Q}\times\mathbb{R}$

Exercise 23. Let $f=1+X+X^2+\cdots+X^{p-1}$, where p is prime, and consider the ring $R=\frac{\mathbb{Q}[X]}{(f^p)}$.

a) $\overline{X^p-p}$ is nilpotent and $\overline{X^p-1}$ is a unit in R

b) $\overline{X^p-p}$ and $\overline{X^p-1}$ are units in R

c) $\overline{X^p-p}$ and $\overline{X^p-1}$ are nilpotent in R

d) Neither $\overline{X^p-p}$ nor $\overline{X^p-1}$ is a unit or nilpotent in R

e) None of these statements are true

Exercise 24. Let R be a commutative ring with the property: $\forall\, r\in R\ \exists\, s\in R$ such that $r=sr^2$. Let I be a finitely generated ideal of R. Then:

a) I is prime

b) I is principal

c) $I=(0)$ or $I=R$

d) $I^2=0$

e) The quotient ring $\frac{R}{I}$ has one unique maximal ideal