

King Fahd University of Petroleum and Minerals,
Department of Mathematics and Statistics- Term 211
Major Exam 2: Math 325, Linear Algebra
Duration: 150 minutes

NAME :

ID :

Solve the following Exercises.

Exercise 1. (5-5 points)

Let W be the subspace of $V = \mathbb{R}^3$ given by $W = \text{span}\{(1, 2, 1), (1, 1, 1)\}$ and consider the scalar product on V defined by $(x_1, x_2, x_3)|(y_1, y_2, y_3) = x_1y_1 - x_2y_2 - x_3y_3$.

(1) Find an orthonormal basis of W .

(2) Set $V = \mathbb{C}^2$ as a vector space over \mathbb{C} with the product scalar defined by:

$(u|v) = x_1y_1 - ix_2y_1 - ix_1y_2 - 2x_2y_2$, $u = (x_1, x_2)$ and $v = (y_1, y_2)$. Find an othogonal basis of V .

Exercise 2. (5-5-5-5 points)

Let V be an n -dimensional vector space over a field K with a non-degenerate scalar product $(|)$, W a subspace of V and V^* its dual space.

(1) Prove that $\dim V = \dim(W) + \dim W^\perp$.

(2) Prove that for any linear functional $T \in V^*$, there is a unique $w \in V$ such that $Tv = (v|w)$ for every $v \in V$.

Assume that $V = \mathbb{R}^3$ with the scalar product $(X|Y) = x_1y_1 + 2x_2y_2 + 4x_3y_3$, and let $B = \{v_1 = (0, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1)\}$ and B^* its dual basis.

(3) Express $v_i^*(x, y, z)$ for each $i = 1, 2, 3$.

(4) Let $T : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by $T(x, y, z) = x + y - z$. Find $w \in \mathbb{R}^3$ such that $Tv = (v|w)$ for every $v \in \mathbb{R}^3$.

Exercise 3. (5-5)

(1) Find the bilinear form associated to the quadratic form $q(x, y, z) = 4x^2 + 8xy + 2xz$.

(2) Find the quadratic form associated to the bilinear form:

$$f(X, Y) = x_1y_1 - x_1y_2 + x_1y_3 - x_2y_1 + x_2y_3 - x_3y_1.$$

Exercise 4. (8-5-2 points)

Let V be an n -dimensional ($n \geq 1$) vector space over \mathbb{R} with a scalar product.

(1) Prove that there is an integer r such that for every orthogonal basis $B = \{v_1, \dots, v_n\}$ of V , there are exactly r integers i such that $(v_i|v_i) > 0$.

Assume that $V = \mathbb{R}^3$ with the scalar product defined by:

$$(X|Y) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2.$$

(1) Find an orthogonal basis of V with respect to the defined scalar product.

(2) Find the index of positivity.

Exercise 5. (5-5-5)

Let V and W be finite-dimensional vector spaces over \mathbb{R} both having inner products and let $L : V \rightarrow W$ be a linear map.

- (1) Show that $\text{Im}(L)^\perp = \text{Ker}(L^*)$, where L^* is the adjoint of L .
- (2) Show that $\dim(\text{Im}(L)) = \dim(\text{Im}(L^*))$.
- (3) Show that the equation $Lx = y$ has a solution if and only if y is orthogonal to the kernel of L^* .

Exercise 6. (5-5-5-5)

Let V be an n -dimensional vector space over \mathbb{R} with a positive definite scalar product (\cdot, \cdot) , T be a linear operator on V and T^* the operator adjoint of T .

- (1) Show that both T^*T and TT^* are self-adjoint.
- (2) Show that $\ker(T) = \ker(T^*T)$.
- (3) If T is one-to-one, show that T^* and T^*T are invertible and find their inverses.
- (4) Show that $\text{Im}(TT^*) = (\ker(TT^*))^\perp$

Exercise 7. (5-5)

Let $V = \mathbb{R}^n$, T a unitary operator on V and A be matrix representing T in a basis B of V . (1) Find $\det(A)$.

(2) Assume that T is annihilated by the polynomial $f(X) = X^2 - 1$. Is T a symmetric operator? Justify.