

# Math405: Learning From Data

## Final Exam

29th December 2021 at 7:00pm<sup>a</sup>

<sup>a</sup>*slimb@kfupm.edu.sa*  
<sup>b</sup>*Duration: 120 minutes*

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### 1. Cholesky Factorization (5 points)

Perform the Cholesky Factorization for  $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ .

### 2. Gradient Descent (5 points)

Consider the problem where we have to:

minimize  $f(x_1, x_2) = (3x_1 - x_2)^2 + (2x_2 - 1)^2$ , starting from  $(1, 1)$ .

Perform manually one full iteration of the Newton-Raphson method.

### 3. Gram-Schmidt (5 points)

Find the Gram-Schmidt factorization of the matrix  $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ .

### 4. Derivative of $A^{-1}$ (5 points)

For  $A(t) = \begin{pmatrix} 3t & 4t \\ 1 & 1 \end{pmatrix}$ , find  $\frac{\partial A^{-1}}{\partial t}$ .

**5. Generalized Eigenvectors (5 points)**

For  $S = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ ,  $M = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ , and  $\lambda$  a generalized eigenvalue such that  $S = \lambda M$ , find the form of the generalized eigenvectors  $X$  that solve  $SX = \lambda MX$ .

**6. Least Squares (5 points)**

For  $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ , find the least squares solution to  $AX = B$ .

**7. Singular values (5 points)**

For  $A = \begin{pmatrix} 2t & 0 \\ -1 & 1 \end{pmatrix}$ , find the singular values  $\sigma_1(t)$  and  $\sigma_2(t)$ .

**8. Matrix Completion (5 points)**

a) Find the matrix  $S = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ , with the smallest  $l_2$  norm  $\|S\|_2$  and

such that:

$$S \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

b) Find the best possible real-valued matrix  $A^*$  that completes  $A = \begin{pmatrix} * & 1 \\ 4 & * \end{pmatrix}$

and minimizes:

$$\min_{C \text{ and } R} \frac{1}{2} \|(A - CR^t)_{\text{known}}\|_2^2 + \frac{1}{2} (\|C\|_F^2 + \|R\|_F^2)$$

where  $CR^t = (2 \times 1)(1 \times 2)$ .