

King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 437 - Final Exam
AY 2022-2023 (Term 221)
Time Allowed: 180 Minutes

Name: ID number:

- Textbook, notes, mobiles and smart devices are not allowed in this exam.
 - Write neatly and legibly. You may lose points for messy work.
 - Show all your work. No points for answers without justification.
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| Question | Marks | Max Marks |
|----------|-------|-----------|
| 1 | | 20 |
| 2 | | 25 |
| 3 | | 20 |
| 4 | | 25 |
| 5 | | 20 |
| 6 | | 20 |
| 7 | | 10 |
| Total | | 140 |

Question 1

Solve the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} = \sqrt{3}y, & \text{for } 0 < x < \pi, 0 < y < 2\pi \\ u(x, 0) = u(x, 2\pi) = 0, & \text{for } 0 < x < \pi \\ u(0, y) = u(\pi, y) = 0, & \text{for } 0 < y < 2\pi. \end{cases}$$

Question 2

Consider the heat equation on the real line

$$\begin{cases} u_t = ku_{xx}, & \text{for } -\infty < x < \infty, t > 0 \\ u(x, 0) = f(x), & \text{for } -\infty < x < \infty. \end{cases} \quad (1)$$

- (a) Derive a solution for (1) using separation of variables and Fourier integral method.
(b) Reformulate the solution and verify that:

$$u(x, t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4kt} d\xi$$

- (c) Write the solution of (1) given that

$$f(x) = \begin{cases} 1, & \text{for } -1 \leq x \leq 1, \\ 0, & \text{for } |x| > 1. \end{cases}$$

Then, expand the integrand of this solution in a Maclaurin series and integrate term by term to obtain a series solution for $u(x, t)$.

Question 3

Solve the Dirichlet problem for the upper half-plane

$$\begin{cases} \nabla^2 u = 0, & \text{for } -\infty < x < \infty, y > 0 \\ u(x, 0) = f(x), & \text{for } -\infty < x < \infty. \end{cases}$$

given that

$$f(x) = \begin{cases} \cos(x), & \text{for } -\pi/2 \leq x \leq \pi/2, \\ 0, & \text{for } |x| > \pi/2. \end{cases}$$

Question 4

Consider the linear first-order partial differential equation:

$$3yu_x - 2xu_y = 0.$$

Determine the characteristics and the general solution of the partial differential equation. For each set of Cauchy data given, attempt to find a particular solution of the Cauchy problem. Determine if there is a unique solution, no solution, or infinitely many solutions.

- (a) $u(x, y) = x^2$ on the line $y = x$.
(b) $u(x, y) = 1 - x^2$ on the line $y = -x$.
(c) $u(x, y) = 2x$ on the ellipse $2x^2 + 3y^2 = 4$.

Question 5

Find the solution of the quasilinear equation

$$yu_x - xu_y = e^u$$

that contains the curve Γ given by $y = \sin x, u = 0$

Question 6

Consider the initial boundary value problem (IBVP):

$$\begin{cases} u_t = 7u_{xx}, & \text{for } 0 < x < 5, t > 0 \\ u(0, t) = 1, \quad u(5, t) = 4, & \text{for } t > 0 \\ u(x, 0) = e^{-x}, & \text{for } 0 < x < 5. \end{cases} \quad (2)$$

Assume $u(x, t) = U(x, t) + \psi(x)$, where

$$U(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{5}\right) e^{-7n^2\pi^2 t/25}$$

satisfies the heat equation $U_t = 7U_{xx}$ for $0 < x < 5, t > 0$ and $U(0, t) = U(5, t) = 0$.

- Solve the IBVP (2) by finding c_n and $\psi(x)$.
- What does the solution $u(x, t)$ represent? Graph the solution for two different values of t . Specify the transient part and the steady-state part of the solution.

Question 7

Prove Duhamel's principle for the wave equation that is the solution of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x, t), & \text{for } -\infty < x < \infty, t > 0 \\ u(x, 0) = u_t(x, 0) = 0, & \text{for } -\infty < x < \infty. \end{cases}$$

is

$$u(x, t) = \int_0^t w(x, t, \tau) d\tau$$

where $w = w(x, t, \tau)$ satisfies

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & \text{for } -\infty < x < \infty, t > \tau \\ w(x, \tau, \tau) = 0, \quad w_t(x, \tau, \tau) = h(x, \tau), & \text{for } -\infty < x < \infty. \end{cases}$$