KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics and Statistics

Math 513 Final Exam

Semester (211)

Dec. 26, 2021

Name: ....KEY.................................

I.D: .... **Section:** ....**Ser:** ....

Instructions

1. No electronic device (such as programmable calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/15
2	/15
3	/20
4	/20
5	/20
Total	/90

Question 1 (15 points)

Use Fourier transform to solve the Laplace equation $u_{xx} + u_{yy} = 0$ over the upper half plane $0 < y$.

We require that the solution remains bounded over the entire domain and specify it along the x-axis,

$$u(x,0) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

(Hint: Use *Poisson's integral formula*)

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(t)}{(x-t)^2 + y^2}\, dt$$

$$= \frac{1}{\pi} \left[\int_0^{\infty} \frac{y}{(x-t)^2 + y^2}\, dt - \int_{-\infty}^{0} \frac{y}{(x-t)^2 + y^2}\, dt \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\infty} \frac{\frac{1}{y}}{\left(\frac{x-t}{y}\right)^2 + 1}\, dt - \int_{-\infty}^{0} \frac{\frac{1}{y}}{\left(\frac{x-t}{y}\right)^2 + 1}\, dt \right]$$

Let $w = \frac{x-t}{y} \Rightarrow dw = \frac{-1}{y}\, dt$

$$\Rightarrow \int_0^{\infty} \frac{\frac{1}{y}}{\left(\frac{x-t}{y}\right)^2 + 1}\, dt = \int_{x/y}^{-\infty} \frac{-dw}{w^2+1} = -\tan^{-1}(w) \Big|_{x/y}^{-\infty} = \frac{+\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)$$

and $$\int_{-\infty}^{0} \frac{\frac{1}{y}}{\left(\frac{x-t}{y}\right)^2 + 1}\, dt = -\tan^{-1}(w) \Big|_{\infty}^{x/y} = -\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2}$$

Thus $$u(x,y) = \frac{1}{\pi} \left[+\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2} \right]$$

$$= \frac{+2}{\pi} \tan^{-1}\left(\frac{x}{y}\right)$$

Question 2 (15 points)

Find the particular solutions for the following differential equations.

$$y'' + 3y' + 2y = e^{-t}H(t), \qquad -\infty < t < \infty.$$

Taking Fourier for bothsides

$$(2 + 3i\omega - \omega^2)\, Y(\omega) = \frac{1}{1+i\omega} \Rightarrow Y(\omega) = \frac{1}{(1+\omega i)^2 + (2+i\omega)}$$

$$y(t) = -\frac{1}{2\pi} \oint \frac{e^{itz}}{i(z-i)^2(z-2i)}\, dz$$

singularities are i and $2i$ (in the upper half)

So, for $t < 0 \Rightarrow y(t) = 0$

for $t > 0$, $y(t) = \frac{-1}{2\pi}(2\pi i)\left\{ Res(f, i) + Res(f, 2i) \right\}$

$$Res(f, i) = Res\left[\frac{e^{itz}}{i(z-i)^2(z-2i)}, i\right] = \lim \frac{d}{dz}\left[\frac{e^{itz}}{i(z-2i)}\right] = -\frac{t-1}{i} e^{-t}$$

$$Res(f, 2i) = Res\left[\frac{e^{itz}}{i(z-i)^2}\right] = -\frac{e^{-2t}}{i}$$

Thus, $y(t) = (t-1)e^{-t} + e^{-2t}, \quad t > 0$

$$= \left[(t-1)e^{-t} + e^{-2t}\right] H(t).$$

Question 3 (20 points)

Solve the following integral equation: $x(t) + t = \frac{1}{\sqrt{\pi}} \int_0^t x'(\tau)(t-\tau)^{-1/2}\, d\tau, \;\; x(0) = 0.$

Take Laplace for both sides

$$X(s) + \frac{1}{s^2} = \frac{1}{\sqrt{\pi}} \left[s X(s) \right] \sqrt{\frac{\pi}{s}}$$

$$\Rightarrow X(s) = -\frac{1}{s^2(1-\sqrt{s})} = \frac{-(1+\sqrt{s})}{s^2(1-s)}$$

$$X(s) = -(1+\sqrt{s})\left(\frac{1}{s^2} + \frac{1}{s} - \frac{1}{s-1}\right)$$

$$x(t) = -t - 1 + e^t - 2\sqrt{\frac{t}{\pi}} + \cancel{\frac{1}{\sqrt{\pi t}}}\; e^t \operatorname{erf}(\sqrt{t})$$

$$= e^t\left[1 + \operatorname{erf}(\sqrt{t})\right] - t - 1 - 2\sqrt{\frac{t}{\pi}}.$$

Question 4

(20 points)

Solve the following system of ordinary differential equations:

$$x' + 2x - y' = 0, \quad x' + y + x = t^2, \quad x(0) = y(0) = 0.$$

First: $sX(s) - x(0) + 2X(s) - sY(s) - y(0) = 0$

Second: $sX(s) - x(0) + Y(s) + X(s) = \frac{2}{s^3}$

$\xrightarrow{\text{solve}}$

$$X(s) = \frac{2}{s^2(s^2+2s+2)} = \frac{1}{s^2} - \frac{1}{s} + \frac{s+1}{(s+1)^2+1}$$

$$Y(s) = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{(s+1)^2+1}$$

$\xrightarrow{\text{inverse}}$

$$x(t) = t - 1 + e^{-t}\cos t$$

$$y(t) = t^2 - t + e^{-t}\sin t$$

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Question 5 (20 points)

In terms of error function, use the Laplace transform to solve the heat equation:

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \ t > 0,$$

subject to: $$u(0,t) = 0, \quad u(1,t) = e, \quad t > 0,$$
$$u(x,0) = 0, \quad 0 < x < 1.$$

$$U_{xx} - sU = 0$$

$$U(0,s) = 0, \quad U(1,s) = \frac{e}{s}$$

$$U(x,s) = c_1 \cosh(x\sqrt{s}) + c_2 \sinh(x\sqrt{s})$$

From BC: $c_1 = 0, \quad c_2 = \dfrac{e}{s \sinh\sqrt{s}}$

Thus $$U(x,s) = \frac{e}{s}\,\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}$$

Using, $$\frac{\sinh x\sqrt{s}}{s\sinh\sqrt{s}} = \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s(1-e^{2\sqrt{s}})}$$

and $$\frac{1}{1-e^{2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}}$$

we have $$\frac{\sinh(\sqrt{s}x)}{s\sinh\sqrt{s}} = \sum_{n=0}^{\infty}\left[\frac{e^{-(2n+1-x)\sqrt{s}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s}}}{s}\right]$$

$$u(x,t) = e\,\mathcal{L}^{-1}\left\{\frac{\sinh(\sqrt{s}x)}{s\sinh\sqrt{s}}\right\}$$

$$= e\sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]$$

$$= e\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]$$

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