	College of Computing and Mathematics - Mathematics Department	
	Midterm Exam	INSTRUCTOR: DR. KAREEM ELGINDY 2022 - Term 221
	Math 513: Mathematical Methods for Engineers	Duration: 120 minutes

Solution Manual

Instructions: This exam consists of two parts. The first part consists of one **MANDATORY** question while the second part consists of four questions. You are required to answer **Part A QUESTION** and **ANY TWO QUESTIONS from Part B**.

Part A Answer the Following MANDATORY Question:

1. Consider the periodic function

$$f(t) = t, \quad \forall -\pi \le t \le \pi, \quad \text{and } f(t + 2\pi) = f(t), \quad \forall t \in \mathbb{R}.$$

(a) **[15 marks]** Determine Fourier series $S(t)$ of f.

(b) **[15 marks]** Can we find Fourier series of f' through term-by-term differentiation of $S(t)$ in light of Fourier Differentiation Property? Explain.

Solution. (a) Here $\underbrace{L = \pi}_{2 \text{ marks}}$ and $\underbrace{\omega_n = n}_{2 \text{ marks}}$. Since f is an odd function on $[-\pi, \pi]$, Fourier coefficients $\underbrace{a_n = 0}_{2 \text{ marks}}$ $\forall n = 0, 1, \ldots$. Also,

$$\underbrace{b_n = \frac{2}{L}\int_0^L f(t)\sin(\omega_n t)\,dt}_{3 \text{ marks}} = \underbrace{\frac{-2(-1)^n}{n}}_{3 \text{ marks}}, \quad \forall n \in \mathbb{Z}^+.$$

Hence, Fourier series of f is given by

$$S(t) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nt), \quad \forall t \in \mathbb{R}. \qquad \boxed{3 \text{ marks}}$$

(b) Notice that $\underbrace{f \text{ is continuous on } (-\pi, \pi)}_{3 \text{ marks}}$ and $\underbrace{f'(t) = 1 \text{ is piecewise continuous on } (-\pi, \pi)}_{3 \text{ marks}}$, but $\underbrace{f(-\pi) = -\pi \ne \pi = f(\pi)}_{6 \text{ marks}}$. Therefore, there is no guarantee to obtain Fourier series of f' through term-by-term differentiation of $S(t)$ in light of Fourier Differentiation Property. $\boxed{3 \text{ marks}}$

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Part B Answer **ANY TWO QUESTIONS from the Following Four Questions:**

1. **[35 marks]** Find Fourier sine series for the function

$$f(t) = e^{kt}, \quad 0 < t < \pi,$$

where $k \in \mathbb{R}^+$.

Solution. Here $\underbrace{L = \pi}_{\text{5 marks}}$ and $\underbrace{\omega_n = n}_{\text{5 marks}}$. For the half-range sine expansion of f, the coefficients are

$$a_n = 0, \quad n = 0, 1, \ldots, \qquad \boxed{\text{5 marks}}$$

$$b_n = \underbrace{\frac{2}{\pi}\int_0^\pi e^{kt}\sin(nt)\,dt}_{\text{5 marks}} = \frac{2}{\pi}\left[-\frac{1}{n}e^{kt}\cos nt + \frac{k}{n^2}e^{kt}\sin nt\right]_0^\pi - \frac{k^2}{n^2}b_n \qquad \boxed{\text{5 marks}}$$

$$\Rightarrow \frac{n^2 + k^2}{n^2}b_n = \frac{2}{n\pi}\left[1 - (-1)^n e^{k\pi}\right]$$

$$\Rightarrow b_n = \frac{2n\left[1 - (-1)^n e^{k\pi}\right]}{\pi(n^2 + k^2)}, \quad n = 1, 2, \ldots. \qquad \boxed{\text{5 marks}}$$

The half-range sine expansion is then

$$S(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\left[1 - (-1)^n e^{k\pi}\right]}{n^2 + k^2}\sin(nt), \quad \forall\, 0 < t < \pi. \qquad \boxed{\text{5 marks}}$$

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2. **[35 marks]** Given that Fourier series of some function f defined on the interval $[-4\pi, 4\pi]$ is

$$S(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left[(2n-1)t\right]}{(2n-1)^2}, \quad \forall -4\pi \le t \le 4\pi.$$

Find the sine phase angle form of Fourier series.

Solution.

Since $\underbrace{a_n = -\frac{4}{\pi(2n-1)^2}}_{\text{6 marks}}$ and $\underbrace{b_n = 0}_{\text{6 marks}}$, $\forall n \in \mathbb{Z}^+$, then

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\omega_n t + \varphi_n\right), \qquad \boxed{\text{5 marks}}$$

with

$$\underbrace{\omega_n = 2n - 1}_{\text{6 marks}}, \quad \underbrace{A_n = \sqrt{a_n^2 + b_n^2} = |a_n| = \frac{4}{\pi(2n-1)^2}}_{\text{6 marks}}, \quad \text{and}$$

$$\varphi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}. \qquad \boxed{\text{6 marks}}$$

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3. **[35 marks]** Solve the ordinary differential equation $y'' - y = f(t)$ by complex Fourier series if the forcing is given by the 2π-periodic function f defined on the interval $[-\pi, \pi]$ by

$$f(t) = |t|, \quad -\pi \le t \le \pi.$$

Solution. The complementary solution y_c of the differential equation is given by

$$y_c = c_1 e^{-t} + c_2 e^t. \qquad \boxed{4 \text{ marks}}$$

To determine the particular solution y_p we begin by replacing the function $f(t)$ by its complex Fourier series representation

$$|t| = \frac{\pi}{2} - \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2}. \qquad \boxed{6 \text{ marks}}$$

By the method of undetermined coefficients, we guess the particular solution

$$y_p(t) = B_0 + \sum_{n=-\infty}^{\infty} A_n e^{i(2n-1)t}. \qquad \boxed{3 \text{ marks}}$$

Since

$$y_p''(t) = \sum_{n=-\infty}^{\infty} -(2n-1)^2 A_n e^{i(2n-1)t}, \qquad \boxed{3 \text{ marks}}$$

then

$$\sum_{n=-\infty}^{\infty} -(2n-1)^2 A_n e^{i(2n-1)t} - \left(B_0 + \sum_{n=-\infty}^{\infty} A_n e^{i(2n-1)t} \right) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2}, \qquad \boxed{4 \text{ marks}}$$

or

$$-B_0 - \frac{\pi}{2} + \sum_{n=-\infty}^{\infty} \left[\frac{2}{\pi(2n-1)^2} - \left\{ 1 + (2n-1)^2 \right\} A_n \right] e^{i(2n-1)t} = 0. \qquad \boxed{4 \text{ marks}} \qquad (1)$$

Because Eq. (1) must hold true for any time, each term must vanish separately and

$$\underbrace{B_0 = -\frac{\pi}{2}}_{4 \text{ marks}}, \quad \text{and} \quad \underbrace{A_n = \frac{2}{\pi(2n-1)^2 \left[1 + (2n-1)^2 \right]}}_{4 \text{ marks}}, \quad \forall n \in \mathbb{Z}.$$

All of the coefficients A_n are finite; hence, our particular solution is correct. Therefore, the general solution is given by

$$y(t) = c_1 e^{-t} + c_2 e^t - \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2 \left[1 + (2n-1)^2 \right]} e^{i(2n-1)t}, \quad \forall t \in \mathbb{R}. \qquad \boxed{3 \text{ marks}}$$

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4. (a) **[20 marks]** Derive the Fourier transform starting from the complex Fourier series representation of a $2L$-periodic function f.

(b) **[15 marks]** Find the Fourier transform of

$$f(t) = \begin{cases} 2, & |t| < 2, \\ 0, & |t| > 2. \end{cases}$$

Express your answer in terms of the sinc function.

Solution. (a) Consider a $2L$-periodic function $f(t)$ with complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \qquad \boxed{\text{2 marks}} \tag{2a}$$

where

$$c_n = \frac{1}{2L}\mathcal{F}(\omega_n), \qquad \boxed{\text{2 marks}} \tag{2b}$$

$$\mathcal{F}(\omega_n) = \underbrace{\int_{-L}^{L} f(t)e^{-i\omega_n t}dt,}_{\text{2 marks}} \quad \underbrace{\omega_n = \frac{n\pi}{L}}_{\text{2 marks}}.$$

Substituting Eq. (2b) into Eq. (2a) yields

$$f(t) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} \mathcal{F}(\omega_n)e^{i\omega_n t}. \qquad \boxed{\text{2 marks}}$$

Let us now introduce the notation $\underbrace{\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}}_{\text{2 marks}}$. Then,

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathcal{F}(\omega_n)e^{i\omega_n t}\Delta\omega. \qquad \boxed{\text{2 marks}}$$

As $\underbrace{L \to \infty}_{\text{2 marks}}$, the angular frequency approaches a continuous variable ω, and $\Delta\omega$ can be interpreted as the infinitesimal $d\omega$. Therefore, the Fourier transform can be defined as an improper Riemann integral with both limits infinite such that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega t}d\omega, \qquad \boxed{\text{2 marks}}$$

and

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \qquad \boxed{\text{2 marks}}$$

(b) From the definition of the Fourier transform,

$$\mathcal{F}\{f(t)\} = \mathcal{F}(\omega) = \underbrace{\int_{-\infty}^{-2} 0 \cdot e^{-i\omega t} dt}_{\text{2 marks}} + \underbrace{\int_{-2}^{2} 2 \cdot e^{-i\omega t} dt}_{\text{2 marks}} + \underbrace{\int_{2}^{\infty} 0 \cdot e^{-i\omega t} dt}_{\text{2 marks}}$$

$$= \underbrace{\frac{4}{\omega}\frac{e^{2i\omega} - e^{-2i\omega}}{2i}}_{\text{3 marks}} = \underbrace{\frac{4\sin(2\omega)}{\omega}}_{\text{3 marks}} = \underbrace{8\,\mathrm{sinc}(2\omega)}_{\text{3 marks}},$$

where $\mathrm{sinc}(x) = \sin(x)/x$ is the sinc function.

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