MATH 528 MIDTERM EXAM (TERM 211)

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NAME:

1. Let V be the vector space of degree at most 2 polynomials, i.e.
$$V = \{f(x) = a_0 + a_1 x + a_2 x^2 : a_1, a_2, a_3 \in \mathbb{R}\}.$$
Let $T : V \to V$ be the linear transform defined by
$$Tf(x) = xf'(x) + x^2 f''(x).$$
Find the matrix representation $[T]_{\mathcal{B}}$, where $\mathcal{B} = \{1, x, x^2\}$.

2. Find eigenvalues and corresponding eigenvectors of the matrix
$$A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}.$$

3. Check that
$$\alpha(s) = \left(\frac{\cos^{-1} s - s\sqrt{1-s^2}}{2}, \frac{1-s^2}{2} \right)$$
is a unit speed curve. Find $\vec{T}(s)$, $\vec{N}(s)$ and the curvature $\kappa(s)$. (Hint. Use $\frac{d}{ds}(\cos^{-1} s) = -\frac{1}{\sqrt{1-s^2}}$.)

4. Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve
$$\alpha(t) = (6t, 3t^2, t^3).$$

5. Check
$$\mathbf{x}(u, v) := (\sin u \cos v, 2 \sin u \sin v, 4 \cos u)$$
is a parametric surface.

6. Consider the Monge patch
$$\mathbf{x}(u, v) = (u, v, u^2 + v^2).$$
Compute $\mathbf{x}_u$ and $\mathbf{x}_v$ at $(u, v) = (2, 1)$. Find also a and b such that
$$\vec{v} = (-1, 2, 0) = a\mathbf{x}_u + b\mathbf{x}_v$$
at $(u, v) = (2, 1)$. (This means $\vec{v}$ is a tangent vector of the surface at $(2, 1, 5)$.)

#1. Let $f_1 = 1$, $f_2 = x$, $f_3 = x^2$. Then

$$Tf_1 = x f_1' + x^2 f_1'' = 0$$

$$= 0 \cdot f_1 + 0 \cdot f_2 + 0 \cdot f_3 \quad \text{since } f_1' = f_1'' = 0$$

$$Tf_2 = x f_2' + x^2 f_2'' = x \cdot 1 + x^2 \cdot 0$$

$$= 0 \cdot f_1 + 1 \cdot f_2 + 0 \cdot f_3 \quad \text{since } f_2' = 1, \quad f_2'' = 0$$

$$Tf_3 = x f_3' + x^2 f_3'' = x \cdot 2x + x^2 \cdot 2$$

$$= 4x^2 \quad \text{since } f_3' = 2x, \quad f'' = 2$$

$$= 0 \cdot f_1 + 0 \cdot f_2 + 4 f_3$$

$$\Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

#2. $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$ Then the characteristic equation is

$$\lambda^2 - (\mathrm{tr} A)\lambda + \det A = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \quad (\lambda - 5)(\lambda + 2) = 0 \quad \Rightarrow \lambda = 5 \text{ or } \lambda = -2$$

Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be an eigenvector for $\lambda = 5$.

$$\Rightarrow \quad (A - 5I)\vec{v} = 0$$

$$\Rightarrow \quad \left(\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $2a - b = 0$ $\Rightarrow$ $b = 2a$. Letting $a = 1$, $b = 2$

$\Rightarrow$ $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 5$.

If $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ is an eigenvector for $\lambda = -2$, we have

$$(A + 2I)\vec{v} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $c + 3d = 0$ $\Rightarrow$ $c = -3d$. Letting $d = 1$,

we have $c = -3$. Therefore

$\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

#3 $\alpha(s) = \left(\frac{\cos^{-1}s - s\sqrt{1-s^2}}{2}, \frac{1-s^2}{2} \right)$

$$\Rightarrow \alpha'(s) = \frac{1}{2}\left(-\frac{1}{\sqrt{1-s^2}} - \sqrt{1-s^2} + \frac{s^2}{\sqrt{1-s^2}}, \quad -2s \right)$$

$$= \frac{1}{2}\left(-\underbrace{\frac{1-s^2}{\sqrt{1-s^2}}}_{=\sqrt{1-s^2}} - \sqrt{1-s^2}, \quad -2s \right)$$

$$= \frac{1}{2}\left(-2\sqrt{1-s^2}, -2s \right) = \left(-\sqrt{1-s^2}, -s \right)$$

$$\Rightarrow |\alpha'(s)|^2 = \left(-\sqrt{1-s^2}\right)^2 + (-s)^2 = 1-s^2+s^2 = 1.$$

$\Rightarrow$ α is of unit speed.

$$\Rightarrow T(s) = \alpha'(s) = \left(-\sqrt{1-s^2}, -s \right)$$

Then $T'(s) = \left(\frac{s}{\sqrt{1-s^2}}, -1 \right)$

$$\Rightarrow \kappa(s) = |T'(s)| = \sqrt{\left(\frac{s}{\sqrt{1-s^2}}\right)^2 + (-1)^2}$$

$$= \sqrt{\frac{s^2}{1-s^2} + 1} = \frac{1}{\sqrt{1-s^2}}$$

$$\& \quad N(s) = \frac{T'(s)}{|T'(s)|} = \frac{T'(s)}{\kappa(s)}$$

$$= \left(s, -\sqrt{1-s^2} \right)$$

#4. $\alpha(t) = (6t,\ 3t^2,\ t^3)$

$$\Rightarrow \alpha'(t) = (6,\ 6t,\ 3t^2) = 3(2,\ 2t,\ t^2)$$

$$\alpha''(t) = (0,\ 6,\ 6t) = 6(0,\ 1,\ t)$$

$$\alpha'''(t) = 6(0,\ 0,\ 1)$$

$$\alpha'(t) \times \alpha''(t) = 18 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2t & t^2 \\ 0 & 1 & t \end{vmatrix}$$

$$= 18\,(t^2,\ -2t,\ 2)$$

$$\Rightarrow k(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \frac{18\sqrt{t^4+4t^2+4}}{27\,(t^4+4t^2+4)^{3/2}}$$

$$= \frac{2}{3(t^4+4t^2+4)}$$

$$\& \ \tau(t) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2} = -\frac{18 \cdot 6 \cdot 2}{18^2\,(t^4+4t^2+4)}$$

$$= -\frac{2}{3(t^4+4t^2+4)}$$

#5 $\vec{X}(u,v) = (\sin u \cos v,\ 2\sin u \sin v,\ 4\cos u)$

It is obviously differentiable, so we only need to check the regularity condition.

$\vec{X_u} = (\cos u \cos v,\ 2\cos u \sin v,\ -4\sin u)$

$\vec{X_v} = (-\sin u \sin v,\ 2\sin u \cos v,\ 0)$

$\Rightarrow \vec{X_u} \times \vec{X_v} = (8\sin^2 u \cos v,\ 4\sin^2 u \sin v,\ 2\sin u \cos u)$

This vector never vanishes for $0 < u < \pi$.

$\Rightarrow \vec{X}$ is regular.

$\Rightarrow \vec{X}$ is a parametric surface.

#7 $\vec{x}(u,v) = (u, v, u^2+v^2)$. At $(u,v)=(2,1)$

$\vec{x}_u = (1, 0, 2u)\big|_{(2,1)} = (1, 0, 4)$

$\vec{x}_v = (0, 1, 2v)\big|_{(2,1)} = (0, 1, 2)$

$\Rightarrow \vec{v} = (-1, 2, 0)$

$= -1\,(1, 0, 4) + 2\cdot(0, 1, 2)$

$= -\vec{x}_u + 2\vec{x}_v$

$\Rightarrow a = -1$ and $b = 2$.