MATH 531 - Major Exam 2

KFUPM, Department of Mathematics and Statistics

Kroumi Dhaker, Term 211

1 Exercise 1(6+7+7 points)

1. Let $f : [0,1] \longrightarrow \mathbb{R}$ be integrable. Show that

$$\left(\int_0^1 f(t)dt\right)^2 \leq \int_0^1 (f(t))^2\, dt.$$

2. Let $f : [a,b] \longrightarrow \mathbb{R}$ be decreasing. Find $TV(f_{[a,b]})$.

3. Let $f(t) = t^2 - 4t + 7$. Calculate $TV(f_{[1,3]})$.

2 Exercise 2(10+10 points)

Let $\{f_n\}_{n=1}^{\infty} \longrightarrow f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Show that $\{f_n\}_{n=1}^{\infty} \longrightarrow g$ in measure on E if and only if $f = g$ a.e.

3 Exercise 3(9+6 points)

Let f be an absolutely continuous function on $[a,b]$.

1. Show that if $|f'| \leq M$ a.e. on $[a,b]$, then f is Lipschitz on $[a,b]$.
2. Deduce that if $f' = 0$ a.e. on $[a,b]$, then f is constant on $[a,b]$.

4 Exercise 4(15 points)

Let $(f_n)_{n=1}^{\infty}$ be measurable on $[0,1]$. Show that if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable on $[0,1]$, then there exists a constant $C < \infty$ such that $\int_{[0,1]} |f| \leq C$.

Hint: think to divide the interval $[0,1]$.

5 Exercise 5(5+5+5 points)

Suppose that f is finite on $[a, b]$ and is of bounded variation on every interval $[a + \varepsilon, b]$, for $\varepsilon > 0$, with $TV(f_{[a+\varepsilon,b]}) \leq M < +\infty$.

1. Show that $|f(t)| \leq M + |f(b)|$ for all $t \in (a, b)$.
2. Deduce that $TV(f_{[a,b]}) < +\infty$.
3. Is $TV(f_{[a,b]}) \leq M$? If not what additional assumptions will make it so?

6 Exercise 6(5+5+5 points)

Let f be a strictly increasing absolutely continuous function on $[0, 1]$.

1. Show that $m\left(f\left((a, b)\right)\right) = \int_{(a,b)} f'$, for any open interval (a, b) of $[0, 1]$.

2. Let O be an open set of $[0, 1]$. Show that $m\left(f\left(O\right)\right) = \int_O f'$.

3. Let F be a closed set of $[0, 1]$. Deduce that $m\left(f\left(F\right)\right) = \int_F f'$.

Hint: The image of an interval by a continuous function is an interval. Any open set is a countable union of disjoints open intervals.