Department of Mathematics, King Fahd University of Petroleum & Minerals,
Math 533 Midterm Exam, 2022-2023 (221)
Maximum Marks : 90 Duration: 130 minutes

Q1: (a) Show that the real part of any solution $(z+1)^{100} = (z-1)^{100}$ must be zero.

(b) If $Re(z^n) \geq 0$ for every positive integer n, show that z is a non-negative real number.

(c) Identify and sketch $Re(z) < Re(z^2)$.

(d) Compute $|\frac{(1-i)^n}{(2+2i)^n}|$.

Q2: (a) Show from the definition that the functions $x = Re(z)$ and $y = Im(z)$ are not differentiable at any point.

(b) If $f'(z) = 0$ in a domai D, then show that $f(z)$ is constant in D.

OR

(b) Let $|f(z)|$ be constant in a domai D, where $f(z)$ is analytic. Then show that $f(z)$ is constant in D.

(c) Show that if $h(z)$ is a complex-valued harmonic function (solution of Laplace equation) such that $zh(z)$ is also harmonic, then $h(z)$ is analytic.

Q3: (a) Find the radius of convergence for $\sum_{n=1}^{\infty} \frac{3^n}{n^2+4n} z^{2n}$.

(b) Evaluate (i) $\int_{|z|=r} z^(- n)\ |dz|$, (ii) $\int_{|z|=r} \bar{z}^n dz$ and (iii) $\int_C Re(z)dz$, C is the lower half of the circle with radius 4 and centre 0. **(Do not use Cauchy's theorems)**

(c) State and prove the ML-inequality and use it to find the **upper bound** of $\int_0^{2+i} e^{z^2} dz$.

OR

(c) State and prove Cauchy's inequality. Let $f(z) = \dfrac{1}{(1-z)^2}, 0 < R < 1$. Show that $f^{(n)}(0) = (n+1)!$ and $(n+1)! \leq \dfrac{n!}{R^n(1-R)^2}$.

Q4: (a) Suppose that f is analytic on $|z - z_0| \leq R$ for some $R > 0$. For $|z - z_0| < R$, show that

$$\frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(\xi)}{(\xi - z)(\xi - z_0)} d\xi = \begin{cases} f'(z_0), & z = z_0 \\ \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0. \end{cases}$$

(b) Evaluate $\oint_{|z|=1'} \dfrac{dz}{\bar{z} - a}$ for $|a| > 1$ and $|a| < 1$. (**Note: Use Cauchy's theorems)**

(c) Let $f(z)$ be an entire function satisfying that $|f(z)| \leq |z|^2$ for all z. Show that $f(z) \equiv az^2$ for some constant a satisfying $|a| \leq 1$.

(d) Evaluate $\oint_C \left(\dfrac{e^z}{z+3} - 3\bar{z}\right)dz$, **where** $C : |z| = 1$.