King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 535: Functional Analysis Exam 2, Spring Semester 212

Solve any set of problems for 100 points.

Problem 1: (40 points) Prove the following:
(a) Each finite-dimensional subspace o f a normed space is complete.
(b) The closure of a subspace of a normed space is also a subspace.

Problem 2:(20 points) Show that in a Banach space each absolutely convergent vector series is convergent.

Problem 3: (30 points) (a) Define bounded linear operators .
(b) Let U and V be normed spaces and let $A : U \to V$ be a linear operator. Show that A is bounded if and only if there exists a constant $K > 0$ such that $\|Au\|_V \leq K\|u\|_U$ for all $u \in U$.

Problem 4: (30 points) (a) Define continuous linear operators.
(b) Let U and V be normed spaces and let $A : U \to V$ be a continuous linear operator. Show that $N(A) \subseteq U$ is closed, where $N(A)$ is the null space of A.

Problem 5: (40 points) **Prove the Hahn-Banach Theorem**. Let U be a real vector space and let $p : U \to \mathbb{R}$ be a real-valued function with the following properties:
(i) $p(u + v) \leq p(u) + p(v)$ for all vectors $u, v \in U$;
(ii) $p(\alpha u) = \alpha p(u)$ for all vectors $u \in U$ and for all real numbers $\alpha \geq 0$.
Let f be a real-valued linear functional defined on a subspace $M \subset U$ satisfying the inequality $f(u) \leq p(u)$ for all $u \in M$. Then, there exists a linear extension F of the functional f to the space U such that $F(u) \leq p(u)$ for all $u \in U$ and $F(u) = f(u)$for all $u \in M$.

Problem 6: (20 points) Let U be a normed space. If its dual U' is separable, show that U itself is also separable.

Problem 7: (20 points) Prove every reflexive Banach space is weakly sequentially complete.

Good luck
Manal Alotaibi