Final Exam

Duration: 180 minutes

ID:	
NAME:	

- Show your work.
- There are empty pages attached to this exam booklet.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	/140

Problem 1 (20 points)

Let U^* be the algebraic dual of the vector space U. For a nonzero $f \in U^*$, show that there exists a vector $u \in U$ such that $f(u) = 1$.

Problem 2 (20 points)

Let $f : X \to Y$ be a function where (X, d) is a metric space and $(Y, \mathfrak{Y})$ is a topological space. Show that f is continuous if every sequence $\{x_n\} \subset X$ such that $x_n \to x$, the sequence $\{f(x_n)\} \subset Y$ converges to $f(x)$.

Problem 3 (20 points)

Let U be a normed space. Prove that U is a Banach space if and only if every absolutely convergent infinite vector series in U is convergent.

Problem 4 (20 points)

Let U be a normed space. If its topological dual U' is separable, show that U itself is also separable.

Problem 5 (20 points)

Let $\mathbb{C}[x]_{[-1,1]}$ be the normed space of complex-valued polynomials defined on the interval $[-1,1]$ with the norm

$$\|p\| = \sqrt{\int_{-1}^{1} [|x||p(x)|^2 + 3|p'(x)|^2]\, dx}, \quad \text{for all} \quad p \in \mathbb{C}[x]_{[-1,1]}$$

(a) Does this norm generate an inner product on $\mathbb{C}[x]_{[-1,1]}$? If yes find it.

(b) Show that

$$\left| \int_{-1}^{1} \left[|x|^3 p(x) + 6xp'(x) \right] dx \right| \leq \frac{5}{\sqrt{3}} \left\{ \int_{-1}^{1} \left[|x||p(x)|^2 + 3|p'(x)|^2 \right] dx \right\}^{1/2}$$

Problem 6 (20 points)

Let $\mathbb{K}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$. Prove that for each vector $x_0 \in \mathcal{H}$, there exits a unique vector $y_0 \in \mathbb{K}$ such that

$$d(x_0, \mathbb{K}) = \|x_0 - y_0\|.$$

Problem 7 (20 points)

Let $\mathcal{O}$ be an orthonormal set of an inner product space H.

(a) If the linear hull of $\mathcal{O}$ is dense in H, show that $\mathcal{O}$ is complete.

(b) If H is a Hilbert space and $\mathcal{O}$ is complete, show that the linear hull of $\mathcal{O}$ is dense in H.