King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 550 Linear Algebra (Term 211)

Major Exam 2 (Duration = 3 hours)

Problem 1. For each one of the following matrices over $\mathbb{R}$, determine its invariant factors and rational form:

$$(1)\quad A=\begin{pmatrix}2&0&0\\1&1&0\\0&0&1\end{pmatrix}\qquad (2)\quad B=\begin{pmatrix}0&1&0\\-1&1&0\\1&0&1\end{pmatrix}\qquad (3)\quad C=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

Problem 2. Let $A=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\\0&0&0&0\end{pmatrix}$

(1) Reduce $xI-A$ to its Smith normal form.
(2) Use the Smith normal form of A to find its invariant factors.
(3) Find the Jordan form of A.
(4) Let T be a linear operator on $\mathbb{R}^4$ such that A is the matrix associated to T in the standard basis $\{e_1,e_2,e_3,e_4\}$. Find an explicit cyclic decomposition of $\mathbb{R}^4$ under T; namely, find α_1 , $\alpha_2\in\mathbb{R}^4$ and their respective T-annihilators p_1 , p_2 such that $\mathbb{R}^4=Z(\alpha_1,T)\oplus Z(\alpha_2,T)$.

Problem 3. Let $A=\begin{pmatrix}1&0&0\\a&1&0\\b&c&2\end{pmatrix}$ with x , y , $z\in\mathbb{R}$ and let J denote the Jordan form of A.

(1) Assume $a=0$. Find J.
(2) Assume $a\neq 0$. Find J.

Problem 4. Give all possible 7×7 complex matrices A in *rational form* with minimal polynomial $(x+1)^2(x-2)$ and with *four* invariant factors.

Problem 5. Let $A=\begin{pmatrix}0&0&a\\0&1&b\\a&0&0\end{pmatrix}$ where a and b are *nonzero* real numbers.

(1) Reduce $xI-A$ to its Smith normal form.
(2) Use the Smith normal form of A to find its minimal polynomial p_o, and find all values of a and b for which A is diagonalizable.
(3) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1,e_2,e_3\}$. Find the respective T-annihilators of e_1,e_2, and e_3.
(4) Show that T has a cyclic vector; namely, find $\alpha\in\mathbb{R}^3$ such that $\mathbb{R}^3=Z(\alpha,T)$, and give the matrix of T in the basis $S:=\{\alpha,T\alpha,T^2\alpha\}$

Problem 6. Let V be a finite-dimensional vector space over a field F and T a linear operator on V. Let p_o denote the minimal polynomial of T and $V=W_1\oplus\cdots\oplus W_k$ its primary decomposition. Prove the following:

(1) If W is an *invariant* subspace, then $W=(W\cap W_1)\oplus\cdots\oplus(W\cap W_k)$. [Hint: Use projections]
(2) If p_o is *irreducible*, then every *invariant* subspace is T-admissible.
(3) Every *invariant* subspace is T-admissible $\Longleftrightarrow p_o=p_1p_2\cdots p_k$ (i.e., $r_i=1$, $\forall\, i$).
(4) Suppose $F=\mathbb{C}$. Deduce from (3) that: T is diagonalizable $\Leftrightarrow$ Every *invariant* subspace is T-admissible.