King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 212

Exam 1 : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Exercise 1. (5-5-5)

Let $V = \mathbb{R}^3$ and let $B = \{(1,0,1),(1,1,1),(0,0,1)\}$ and $B' = \{(1,0,1),(0,1,0),(-1,0,0)\}$ two bases for V.

(1) Find the transition matrix P from B' to B.

(2) Let T a linear operator on V with $[T]_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}$. Find $[T]_{B'}$.

(3) Is T an isomorphism?

Exercise 2. (7-5-5-3 points)

Let V be an m-dimensional vector space over F and T a linear operator with m distinct characteristic values $c_1 = 1, c_2, \ldots, c_m$ such that $|c_j| < 1$ for $j = 2, \ldots, m$.

(1) Prove that for every vector $u \in V$, $\lim_{n\to+\infty} T^n u$ exists.

We define a linear operator N on V by $Nu = \lim_{n\to+\infty} T^n u$ for every $u \in V$.

(2) Find a basis B_1 for $Nullspace(N)$ and $dim(Nullspace(N))$.

(3) Find a basis B_2 for $range(N)$ and $dim(range(N))$.

(4) Find the matrix representing N in the basis $B = B_1 \cup B_2$.

Exercise 3. (3-7-3-7)

Let V be an n-dimensional vector space over a field F, V^* its dual space, $\{u_1, \ldots, u_n\}$ a basis of V and $B = \{f_1, \ldots, f_n\}$ a basis of V^*.

(1) Prove that $\{u_1^*, \ldots, u_n^*\}$ is a basis of V^*.

(2) Prove that there is a basis S of V such that the dual basis $S^* = B$.

Application: Assume that $V = \mathbb{R}^3$, $B = \{f_1, f_2, f_3\}$ where $f_1(x,y,z) = x + y - z$, $f_2(x,y,z) = x - y + z$ and $f_3(x,y,z) = -x + y + z$ and S_1 the basis of V given by $S_1 = \{(1,1,0),(1,0,1),(0,1,1\}$.

(3) Verify that B is a basis of V^*.

(4) Use the basis S_1 to construct a basis S of V such that $S^* = B$.

Exercise 4. (5-5-5 points)

(1) Let V be the vector space of real-valued continuous functions on $[-a, a]$, $T : V \to V, f \mapsto T(f)$ the linear operator on V defined by $T(f)(x) = \int_0^x f(t)dt$, W_e the subspace of V of even functions and W_o the subspace of V of odd functions. Are W_e and W_o invariant under T? Justify.

(2) Let V be the vector space of all polynomials over the field $\mathbb{R}$. Fix a positive integer $n \geq 1$ and let $\mathbb{P}_n = \{f(X) \in V | deg(f(X)) \leq n\}$. Find a direct summand of $\mathbb{P}_n$, that is a subspace W of V such that $V = \mathbb{P}_n \bigoplus W$.

(3) Use (2) to find an infinite-dimensional vector space V with a linear operator T and two subspaces W_1 and W_2 such that $V = W_1 \bigoplus W_2$, W_1 is a finite-dimensional T-invariant subspace but W_2 is not.

Exercise 5. (5-7-4-4)

Let V be an n-dimensional over a field $\mathbb{F}$ and T a linera operator on V.

(1) Prove that if c_1 and c_2 are **distinct** characteristic values of T with associated characteristic vectors v_1 and v_2 (resp.), then v_1 and v_2 are linearly independent.

(2) Assume that $dim(range(T)) = r$. Prove that T has at most $r + 1$ characteristic values.

(3) Assume that $\mathbb{F} = \mathbb{C}$ and $T^n = I$. Find the minimal and the characteristic polynomial of T.

(4) Is T a diagonalizable operator? Justify.