King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 221

Exam 2 : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Exercises 1:

Exercises 2:

Exercises 3:

Exercises 4:

Total:

Exercise 1. (5-5-5-5)

Let V be a finite-dimensional vector space over a field F and T and H be two linear operators on V.

(1) Prove that if T and H are simultaneously diagonalizable, then T and H commute.

(2) Assume that T and H commute and H has exactly n distinct characteristic values. Prove that there is a basis B such that $[T]_B$ and $[H]_B$ are both diagonal.

Application: Set $V = \mathbb{R}^3$ as a vector space over $\mathbb{R}$, S its standard basis and let $T(x, y, z) = (z, y, x)$ and $H(x, y, z) = (y, x + z, y)$.

(3) Find a basis B of V such that T and H are simultaneously diagonalizable.

(4) Find a matrix P such that $P^{-1}[T]_S P$ and $P^{-1}[H]_S P$ are both diagonal.

Exercise 2. (6-6-6-4-3 points)

Let $V = \mathbb{R}^4$ and T be the linear operator on V defined by:

$T(x, y, z, t) = (2x + y, x + 2y, t, -z + 2t)$. Use the standard basis to:

(1) Find the Smith normal form of $xI - T$ and the invariant factors of T.

(2) Find the cyclic decomposition of $\mathbb{R}^4$ under T.

(3) Find the primary decomposition of $\mathbb{R}^4$ under T.

(4) Find the rational matrix form of T.

(5) Find the Jordan matrix form of T.

Exercise 3. (5-5-5-5-5)

Let V be an n-dimensional $\mathbb{F}$-vector space and T a linear operator on V.

(1) Assume that $\mathbb{F} = \mathbb{C}$ and 0 is the unique characterestic value of T. Prove that T is Nilpotent.

(2) Assume that $\mathbb{F} = \mathbb{R}$. If 0 is the only characteristic value of T, is T Nilpotent? Justify.

(3) Assume that T has a cyclic vector. Prove that any linear operator H commutatindg with T is a polynomial in T.

Application: Set $V = \mathbb{R}^3$ and let T be the linear operator given by $T(x, y, z) = (x + y, z - y, 2z)$.

(4) Find a cyclic vector of T (if it exists).

(5) Prove that for a linear operator U on V, every nonzero vector of V is a cyclic vector of U if and only if the characteristic polynomial of U is irreducible on F.

Exercise 4. (8-3-4 points)

Let $V = \mathbb{R}^3$.

(1) Find the matrix, **in a rational form**, of a linear operator T on V with minimal polynomial $P = (X-1)(X-2)$ and exactly Two invariant factors such that the real-valued function $x \mapsto f(x)$, where $f(X)$ is the characteristic polynomial of T, has an inflection point at $\frac{5}{2}$.

(2) Find the Jordan matrix form of T.

(3) Prove that if every subspace of V is T-invariant, then $T = cI$ for some $c \in F$.