King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 565 Final Exam
The First Semester of 2022-2023 (221)
Time Allowed: 150mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: Consider the nonlinear system

$$\frac{dx}{dt} = (x-y)(x+1)$$
$$\frac{dy}{dt} = (x+y)(y-1).$$

1.)(8pts) Find all critical points of the system.
2.)(12pts) Study stability of the critical points of the system.

Solution

1) $\begin{cases} (x-y)(x+1)=0 \\ (x+y)(y-1)=0 \end{cases} \Rightarrow x=y \quad \text{or} \quad x=-1 \Rightarrow (y-1)^2=0,\ y=1$

$\Downarrow$

$2y(y-1)=0,\ y=0,1$

The critical points are the origin, $A\begin{pmatrix}1\\1\end{pmatrix}$ and $B\begin{pmatrix}-1\\1\end{pmatrix}$

2) Stability by linearization

The Jacobian $J(x_0,y_0)=\begin{pmatrix} x_0+1+x_0-y_0 & -(x_0+1) \\ y_0-1 & y_0-1+x_0+y_0 \end{pmatrix}$

At A,

$J=\begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}$, $\begin{vmatrix} 2-\lambda & -2 \\ 0 & 2-\lambda \end{vmatrix}=0$, $(2-\lambda)^2=0$, $\lambda=2,2$

A is unstable

At the origin:

$J=\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, $\begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix}=0$, $-(1-\lambda^2)-1=0$, $\lambda=\pm\sqrt{2}$

The origin is unstable

At B

$J=\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{vmatrix} -2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix}=0$, $\lambda(2+\lambda)=0$, $\lambda=0,-2$

B is stable.

Problem 2: Consider the nonlinear system

$$\frac{dx}{dt} = x + x^2 - xy + y^2 + f(x,y)$$
$$\frac{dy}{dt} = 2x - y + xy.$$

Assume that

$$f(0,0) = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{|x|+|y|} = 0.$$

1.)(14pts) Show that the system is almost linear.
2.)(6pts) Deduce the stability of the origin.

Solution

1.) If $X = \begin{pmatrix} x \\ y \end{pmatrix}$, then $\dot{X} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}}_{A} X + \underbrace{\begin{pmatrix} x^2 - xy + y^2 + f(x,y) \\ xy \end{pmatrix}}_{F(X}$

We have $|F(X)| = |x^2 - xy + y^2 + f(x,y)| + |xy|$

$$\leq x^2 + \underbrace{|xy|}_{\leq \frac{1}{2}x^2 + \frac{1}{2}y^2} + y^2 + |f(x,y)| + \underbrace{|x||y|}_{\leq \frac{1}{2}x^2 + \frac{1}{2}y^2}$$

$$\leq 2(x^2 + y^2) + |f(x,y)|$$

$$\leq 2(|x| + |y|)^2 + |f(x,y)|$$

$$\Rightarrow \quad \frac{|F(X)|}{|X|} \leq 2(|x| + |y|) + \frac{|f(x,y)|}{|x| + |y|}$$

$\Rightarrow \lim\limits_{|X| \to 0} \dfrac{|F(X)|}{|X|} = 0$, since limit of the RHS is zero

2.) $\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = 0$, $-(1-\lambda^2) = 0$, $\lambda = \pm 1$

The origin is unstable.

Problem 3: Consider the linear system

$$\begin{cases} x' = \left(2 + \dfrac{2\cos 2t}{3+\sin 2t}\right) x \\ y' = x - y. \end{cases} \tag{1}$$

1.)(4pts) Write System (1) in the form $X' = A(t)X$, where $X = \begin{pmatrix} x \\ y \end{pmatrix}$. What is the smallest period of $A(t)$?

2.)(8pts) Show that the general solution (1) is

$$x(t) = c_1(3+\sin 2t)e^{2t},$$
$$y(t) = c_1(1 + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t)e^{2t} + c_2 e^{-t}$$

where c_1, c_2 are arbitrary constants.

You may use the relation $\int e^{3t}\sin 2t dt = (\frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t)e^{3t}$.

3.)(8pts) Find the characteristic multipliers ρ_1 and ρ_2 of the system.

Solution

1.) $\dot{X} = \underbrace{\begin{pmatrix} 2 + \frac{2\cos 2t}{3+\sin 2t} & 0 \\ 1 & -1 \end{pmatrix}}_{A(t)} X$. $A(t+\pi) = A(t)$.

A is periodic of period π

2.) $\int \frac{dx}{x} = \int \left(2 + \frac{2\cos 2t}{3+\sin 2t}\right) dt \Rightarrow \ln|x| = 2t + \ln(3+\sin 2t) + C$

$\Rightarrow x(t) = c_1(3+\sin 2t)e^{2t}$

$\frac{dy}{dt} + y = x$, $\frac{d}{dt}(ye^{t}) = xe^{t} = c_1(3+\sin 2t)e^{3t}$

$\Rightarrow ye^{t} = c_1\int(3+\sin 2t)e^{3t} + c_2$

$= c_1\left[e^{3t} + \int \sin 2t\, e^{3t} dt\right] + c_2$

$= c_1\left[1 + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t\right]e^{3t} + c_2$

$\Rightarrow y(t) = c_1\left[1 + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t\right]e^{2t} + c_2 e^{-t}$

3.) $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 3+\sin 2t \\ 1 + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t \end{pmatrix} e^{2t}}_{X_1} + c_2 \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}}_{X_2}$

X_1 and X_2 are solutions of the system.

$\lambda_1 = 2$, $\lambda_2 = -1$ are characteristic exponents

$\rho_1 = e^{2\pi}$, $\rho_2 = e^{-\pi}$ are characteristic multipliers

Problem 4: Consider the linear system

$$\frac{dx}{dt} = -x - 4y$$
$$\frac{dy}{dt} = 4x.$$

Let $V(x,y) = x^2 + y^2$ and $W(x,y) = x^2 + \varepsilon xy + y^2$

1.)(6pts) Compute $\frac{dV}{dt}$. Deduce the stability of the origin.

2.)(14pts) Compute $\frac{dW}{dt}$. Find a value of $\varepsilon > 0$ such that $\frac{dW}{dt}$ is negative definite and W is positive definite on $\mathbb{R}^2$. Deduce the stability of the origin.

Solution

1.) $\frac{dV}{dt} = 2x\dot{x} + 2y\dot{y} = 2x(-x-4y) + 2y(4x) = -2x^2 = V^*$

V is positive definite on $\mathbb{R}^2$, V^* is negative

$\Rightarrow$ the origin is stable.

2.) $\frac{dW}{dt} = 2x\dot{x} + \varepsilon(x\dot{y} + \dot{x}y) + 2y\dot{y}$

$= 2x(-x-4y) + \varepsilon[4x^2 + y(-x-4y)] + 8yx$

$= -(2-4\varepsilon)x^2 - \varepsilon xy - 4\varepsilon y^2$

$\Delta = \varepsilon^2 - 16\varepsilon(2-4\varepsilon)$

$= \varepsilon[\varepsilon - 16(2-4\varepsilon)] = \varepsilon[65\varepsilon - 32]$

if $\varepsilon \in (0, \frac{32}{65})$, then $\Delta < 0$, $\frac{dW}{dt} < 0$

For W, $\Delta = \varepsilon^2 - 4$,

If $\varepsilon \in (0,2)$, then $W > 0$

Conclusion: If we choose $\varepsilon \in (0, \frac{32}{65})$, then W is positive definite and $\frac{dW}{dt}$ is negative definite on $\mathbb{R}^2$, thus the origin is globally asymptotically stable.

Problem 5:

1.)(10pts)Use Lyapunov second method to show that the origin is unstable for the nonlinear system

$$\frac{dx}{dt} = x^3 - y^3$$
$$\frac{dy}{dt} = 3xy^2 + 6x^2y + 3y^3.$$

2.)(10pts)Use Lyapunov second method to show that the origin is globally asymptotically stable for the nonlinear system

$$\frac{dx}{dt} = -3x - 2xy^2$$
$$\frac{dy}{dt} = -2y - x^2y.$$

Solution

1) $$\begin{cases} \dfrac{dx}{dt} = x^3 - y^3 & \times x \\ \dfrac{dy}{dt} = 3xy^2 + 6x^2y + 3y^3 & \times y \end{cases} \Rightarrow \begin{cases} \dfrac{1}{2}\dfrac{d}{dt}x^2 = x^4 - y^3x \\ \dfrac{1}{2}\dfrac{d}{dt}y^2 = 3xy^3 + 6x^2y^2 + 3y^4 \end{cases}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\underbrace{(3x^2+y^2)}_{\substack{\text{positive definite} \\ \text{on } \mathbb{R}^2}} = 3x^4 + 6x^2y^2 + 3y^4 = 3\underbrace{\overbrace{(x + 2x^2y^2 + y^4)}^{(x^2+y^2)^2}}_{\substack{\text{positive definite} \\ \text{on } \mathbb{R}^2}}$$

$\Rightarrow$ the Origin is unstable

2.) $$\begin{cases} \dfrac{dx}{dt} = -3x - 2xy^2 & \times x \\ \dfrac{dy}{dt} = -2y - x^2y & \times y \end{cases} \Rightarrow \begin{cases} \dfrac{1}{2}\dfrac{d}{dt}x^2 = -3x^2 - 2x^2y^2 \\ \dfrac{1}{2}\dfrac{d}{dt}y^2 = -2y^2 - x^2y^2 \end{cases}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\underbrace{(x^2+y^2)}_{\substack{\text{positive} \\ \text{definite} \\ \text{on } \mathbb{R}^2}} = \underbrace{\underbrace{-3x^2 - 2y^2}_{\substack{\text{negative} \\ \text{definite on } \mathbb{R}^2}} \underbrace{- 3x^2y^2}_{\text{negative on } \mathbb{R}^2}}_{\text{negative definite on } \mathbb{R}^2}$$

$\Rightarrow$ The origin is globally asymptotically stable.