



Solution Manual

Answer ANY SIX QUESTIONS from the Following Seven Questions:

1. (a) [8 marks] Prove that the polyhedral set

$$S = \{(x_1, x_2, x_3)^t \in \mathbb{R}^3 : x_1 + 5x_2 - 2x_3 \leq 0, \quad 4x_1 - 7x_2 + 10x_3 \leq -6\},$$

is a convex set.

*Solution.* We can write  $S = \{x \in \mathbb{R}^3 : Ax \leq b\}$ , where  $x = (x_1, x_2, x_3)^t$ ,  $A = \begin{pmatrix} 1 & 5 & -2 \\ 4 & -7 & 10 \end{pmatrix}$ , and

$b = (0, -6)^t$ . To prove that  $S$  is a convex set, let  $x^{(1)}, x^{(2)} \in S$  and define  $z = \lambda x^{(1)} + (1 - \lambda)x^{(2)}$ , for some  $\lambda \in [0, 1]$ . Then  $Az = A[\lambda x^{(1)} + (1 - \lambda)x^{(2)}] = \lambda Ax^{(1)} + (1 - \lambda)Ax^{(2)} \leq \lambda b + (1 - \lambda)b = b$ . Hence,  $z \in S$  and  $S$  is convex.

(b) [7 marks] Investigate the existence of a minimizing solution to the optimization problem

$$\begin{aligned} \min_{x, y, z} \quad & x^4 - e^{2y} + \cos(z^4) \\ \text{s.t.} \quad & x^6 + y^6 + z^6 \leq 1. \end{aligned}$$

*Solution.* Notice that the feasible set  $S = \{(x, y, z)^t : x^6 + y^6 + z^6 \leq 1\}$  is nonempty, since  $(0, 0, 0)^t \in S$ .  $S$  is closed, since  $\partial S = \{(x, y, z)^t : x^6 + y^6 + z^6 = 1\} \subset S$ . Also,  $S$  is bounded, since

$$x^6 + y^6 + z^6 \leq 1 \Rightarrow x, y, z \leq 1 \Rightarrow x^t x = x^2 + y^2 + z^2 \leq 3 < \infty.$$

Therefore,  $S$  is compact. Moreover, the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , defined by  $f(x) = x^4 - e^{2y} + \cos(z^4)$ , is continuous on  $\mathbb{R}^3$ , since it is the sum of continuous functions. Hence, the conditions of the Weierstrass theorem are satisfied, and the problem admits at least one minimizing solution.

2. Consider the following minimization problem:

$$\min_{x_1, x_2} \quad (x_1 - 1)^2 + (x_2 + 1)^2 \tag{1a}$$

$$\text{s.t.} \quad x_1 + x_2 \leq 3, \tag{1b}$$

$$4x_1 + x_2 \leq 5, \tag{1c}$$

$$x_1 \geq 0, \tag{1d}$$

$$x_2 \geq 0. \tag{1e}$$

(a) [8 marks] Write a necessary and sufficient condition for optimality of the problem.

*Solution.* Notice that the feasible set  $S$  can be defined as  $S = \{x \in \mathbb{R}^2 : Ax \leq b\}$ , where  $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$

and  $b = (3, 5, 0, 0)^t$ . Thus,  $S$  is a polyhedral set, which is convex. Observe also that

$$H(\bar{x}) = \begin{bmatrix} \frac{\partial^2 f(\bar{x})}{\partial x_1^2} & \frac{\partial^2 f(\bar{x})}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(\bar{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\bar{x})}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \forall \bar{x} \in S,$$

which is PD everywhere in  $S$ . Therefore,  $f$  is strictly convex on  $S$  and the problem is a convex programming problem. Since,  $f$  is differentiable on  $S$ , then a necessary and sufficient condition for  $\bar{x}$  to be an optimal solution for the problem is that  $[\nabla f(\bar{x})]^t (x - \bar{x}) \geq 0, \quad \forall x \in S$ .

(b) [7 marks] Are  $(1, 0)^t$  and  $(0, 1)^t$  optimal solutions? Why?

*Solution.* At  $\bar{\mathbf{x}} = (1, 0)^t$  we have  $\nabla f(\bar{\mathbf{x}}) = \left[ \frac{\partial f(\bar{\mathbf{x}})}{\partial x_1}, \frac{\partial f(\bar{\mathbf{x}})}{\partial x_2} \right]^t = [2(1-1), 2(0+1)]^t = [0, 2]^t$ . Therefore,

$$[\nabla f(\bar{\mathbf{x}})]^t (\mathbf{x} - \bar{\mathbf{x}}) = [0, 2] [\mathbf{x} - (1, 0)^t] = [0, 2] (x_1 - 1, x_2)^t = 2x_2 \geq 0.$$

Therefore,  $\bar{\mathbf{x}}$  satisfies the necessary and sufficient optimality condition; hence, it is the unique, strong global optimal solution for the optimization problem. Consequently, the point  $\hat{\mathbf{x}} = (0, 1)^t$  cannot be an optimal solution.

3. (a) [8 marks] Check the quasiconvexity of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(\mathbf{x}) = x_1 + x_2^3$ .

*Solution.* Let  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbb{R}^2 : f(\mathbf{x}^{(1)}) \leq f(\mathbf{x}^{(2)}) \Rightarrow x_1^{(1)} + [x_2^{(1)}]^3 \leq x_1^{(2)} + [x_2^{(2)}]^3$ . Also,

$$[\nabla f(\mathbf{x}^{(2)})]^t [\mathbf{x}^{(1)} - \mathbf{x}^{(2)}] = \left( 1, 3[x_2^{(2)}]^2 \right) [x_1^{(1)} - x_1^{(2)}, x_2^{(1)} - x_2^{(2)}]^t = [x_1^{(1)} - x_1^{(2)}] + 3[x_2^{(2)}]^2 [x_2^{(1)} - x_2^{(2)}] \not\leq 0,$$

so  $f$  is not quasiconvex. For example, let  $\mathbf{x}^{(1)} = (2, -2)^t, \mathbf{x}^{(2)} = (1, 0)^t$ . Note that  $f(\mathbf{x}^{(1)}) = -6$  and  $f(\mathbf{x}^{(2)}) = 1$ , so that  $f(\mathbf{x}^{(1)}) < f(\mathbf{x}^{(2)})$ . But  $[\nabla f(\mathbf{x}^{(2)})]^t [\mathbf{x}^{(1)} - \mathbf{x}^{(2)}] = (2-1) + 3(0)^2(-2-0) = 1 > 0$ .

(b) [7 marks] Which of the plots shown in Figure 1 below represent(s) the graph(s) of a function that is strictly quasiconvex but is neither quasiconvex nor convex. Why? [Explain the reason only when the function satisfies the foregoing condition. Otherwise, state only the type(s) of the function from the three possible types, namely, strictly quasiconvex, quasiconvex, or convex.]

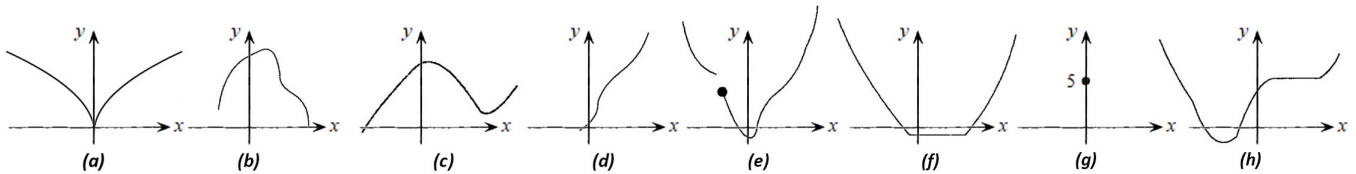


Figure 1: Plots of eight various functions.

*Solution.* Let  $f_i, i = 1, \dots, 8$  denote the function represented by the plots (a) through (h), respectively. Notice that  $f_1, f_4$ , and  $f_5$  are strictly quasiconvex and quasiconvex but not convex.  $f_2$  and  $f_3$  are none of the three types.  $f_6$  is strictly quasiconvex, quasiconvex, and convex.  $f_8$  is quasiconvex but neither strictly quasiconvex nor convex.  $f_7$  is the only function which is strictly quasiconvex but is neither quasiconvex nor convex. To illustrate further, notice that

$$f_7(\lambda x_1 + (1 - \lambda)x_2) = 0 < 5 = f_7(0), \quad \forall x_1, x_2 \in \mathbb{S} : f_7(x_1) \neq f_7(x_2), \quad \lambda \in (0, 1),$$

so  $f_7$  is strictly quasiconvex. It is not quasiconvex because  $f(x_1) = f(x_2) = 0$  for  $x_1 = -1$  and  $x_2 = 1$ , but

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = f(0) = 5 > 0 = \max\{f(x_1), f(x_2)\}.$$

It is also not convex, since for the same points  $x_1$  and  $x_2$ ,

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = f(0) = 5 > 0 = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

4. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  shown in Figure 2 below and defined by

$$f(x) = \min\{f_1(x), f_2(x)\},$$

where

$$f_1(x) = -|x|, \quad \forall x \in \mathbb{R}, \quad \text{and}$$

$$f_2(x) = 4 - (x - 2)^2, \quad \forall x \in \mathbb{R}.$$

- (a) [10 marks] Find the subgradients of  $f$ .
- (b) [5 marks] Is  $f$  subdifferentiable at  $x = 0, 1$ , and  $5$ ? Explain.

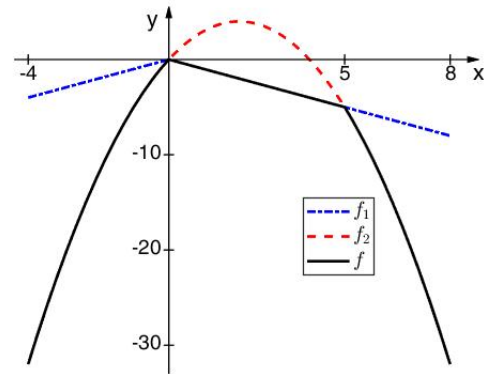


Figure 2: Plots of the functions  $f, f_1$ , and  $f_2$  on the interval  $[-4, 8]$ .

*Solution.* (a) Since  $f_2(x) \geq f_1(x)$ , for  $0 \leq x \leq 5$ ,  $f$  can be redefined by

$$f(x) = \begin{cases} -x, & 0 \leq x \leq 5, \\ 4 - (x - 2)^2, & \text{otherwise.} \end{cases}$$

Note that  $\xi = -1$  is the unique subgradient of  $f$  at any point  $x \in (0, 5)$ .  $\xi = -2(x - 2)$  is the unique subgradient of  $f$  for  $x < 0$  or  $x > 5$ . At  $x = 0$  and  $5$ , the subgradients are not unique because many supporting hyperplanes exist at each point. The family of subgradients at  $x = 0$  and  $5$  are characterized in respective order by

$$\xi|_{x=0} = \lambda \nabla f_1(0) + (1 - \lambda) \nabla f_2(0) = \lambda(-1) + (1 - \lambda)(4) = 4 - 5\lambda, \quad \forall \lambda \in [0, 1],$$

$$\xi|_{x=5} = \lambda \nabla f_1(5) + (1 - \lambda) \nabla f_2(5) = \lambda(-1) + (1 - \lambda)(-6) = -6 + 5\lambda, \quad \forall \lambda \in [0, 1].$$

- (b)  $f$  is subdifferentiable at  $x = 0, 1$ , and  $5$ , since  $\partial f(0) = [-1, 4]$ ,  $\partial f(1) = \{-1\}$ , and  $\partial f(5) = [-6, -1]$ . ▶

5. (a) [10 marks] Show that a polytope  $\mathbb{S} \subseteq \mathbb{R}^n$  is a compact, convex set.

*Solution.* By definition, a polytope is the convex hull of a finite set of points. Let the finite set be  $\mathbb{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ , for some  $k \in \mathbb{R}^+$ . Then,  $\mathbb{X}$  is bounded and closed, since it is a finite union of the singleton sets  $\{\mathbf{x}_i\}, i = 1, \dots, k$ , which are bounded and closed. Therefore,  $\mathbb{X}$  is a compact set; consequently,  $\mathbb{S} = \text{conv}(\mathbb{X})$  is compact. ▶

- (b) [5 marks] Is  $\text{conv}((-1, 10)^t, (5, -1)^t, (0, 0)^t, (-1/2, 1)^t)$  a compact, convex set in  $\mathbb{R}^2$ ? Explain.

*Solution.* Notice that  $\text{conv}((-1, 10)^t, (5, -1)^t, (0, 0)^t, (-1/2, 1)^t)$  is a polytope, since it is the convex hull of the finite set of points  $\mathbb{S} = \{(-1, 10)^t, (5, -1)^t, (0, 0)^t, (-1/2, 1)^t\}$ . By Part (a) of the question,  $\text{conv}((-1, 10)^t, (5, -1)^t, (0, 0)^t, (-1/2, 1)^t)$  is a compact, convex set in  $\mathbb{R}^2$ . ▶

6. Let  $\mathbb{S}$  be a nonempty, closed, convex set in  $\mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^n - \mathbb{S}$ .

- (a) [10 marks] Show that there exists a nonzero vector  $\mathbf{p}$  and a scalar  $\alpha$  such that  $\mathbf{p}^t \mathbf{y} > \alpha$  and  $\mathbf{p}^t \mathbf{x} \leq \alpha$ , for each  $\mathbf{x} \in \mathbb{S}$ .

*Solution.* By the given assumptions, there exists a unique point  $\bar{\mathbf{x}} \in \mathbb{S}$  with minimum distance from  $\mathbf{y}$ . Furthermore,  $\bar{\mathbf{x}}$  is the minimizing point if and only if  $(\mathbf{y} - \bar{\mathbf{x}})^t (\mathbf{x} - \bar{\mathbf{x}}) \leq 0, \forall \mathbf{x} \in \mathbb{S}$ . Now, let  $\mathbf{p} = \mathbf{y} - \bar{\mathbf{x}}$  and  $\alpha = \mathbf{p}^t \bar{\mathbf{x}}$ . Notice that  $\mathbf{p} \neq \mathbf{0}$  and

$$(\mathbf{y} - \bar{\mathbf{x}})^t (\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{p}^t (\mathbf{x} - \bar{\mathbf{x}}) \leq 0, \quad \forall \mathbf{x} \in \mathbb{S} \Rightarrow \mathbf{p}^t \mathbf{x} \leq \mathbf{p}^t \bar{\mathbf{x}} = \alpha, \quad \forall \mathbf{x} \in \mathbb{S}.$$

Also,  $\mathbf{p}^t \mathbf{y} - \alpha = \mathbf{p}^t \mathbf{y} - \mathbf{p}^t \bar{\mathbf{x}} = \mathbf{p}^t (\mathbf{y} - \bar{\mathbf{x}}) = \mathbf{p}^t \mathbf{p} = \|\mathbf{p}\|^2 > 0 \Rightarrow \mathbf{p}^t \mathbf{y} > \alpha, \quad \forall \mathbf{x} \in \mathbb{S}$ . ▶

(b) [5 marks] Show also that  $\alpha = \max\{\mathbf{p}^t \mathbf{x} : \mathbf{x} \in \mathbb{S}\}$ .

*Solution.* By the above answer,  $\exists \bar{\mathbf{x}} \in \mathbb{S} : \mathbf{p}^t \mathbf{x} \leq \mathbf{p}^t \bar{\mathbf{x}} = \alpha \Rightarrow \alpha = \max\{\mathbf{p}^t \mathbf{x} : \mathbf{x} \in \mathbb{S}\}$ . ◀

7. (a) [8 marks] Check the definiteness of the matrix  $\mathbf{H} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}$  using The Superdiagonalization Algorithm.

*Solution.* Let  $\mathbf{H} = (h_{ij})_{1 \leq i, j \leq 3}$ . Clearly,  $\mathbf{H}$  is symmetric since  $\mathbf{H} = \mathbf{H}^t$ . Notice that  $h_{ii} > 0 \forall i$ , so by The Superdiagonalization Algorithm, we apply Gauss elimination to eliminate the off-diagonal elements of the first column:

$$\mathbf{H} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{Gauss}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

Since  $h_{11} = 2 > 0$  and  $\begin{pmatrix} 3/2 & -1 \\ -1 & 4 \end{pmatrix}$  is symmetric and PD, then  $\mathbf{H}$  is PD. ◀

(b) [7 marks] Suppose that  $f : \mathbb{S} \rightarrow \mathbb{R}$  is a twice differentiable function on a nonempty, open, convex set  $\mathbb{S} \subseteq \mathbb{R}^n$ . If the Hessian matrix of  $f$  is the matrix  $\mathbf{H}$  given in Part (a) of the question, then show that  $f$  must be a convex function.

*Solution.* Since  $\mathbf{H}(\bar{\mathbf{x}})$  is PD  $\forall \bar{\mathbf{x}} \in \mathbb{S}$ , then under the given assumptions,  $f$  must be strictly convex; hence,  $f$  is convex. ◀