



Answer ANY SIX QUESTIONS from the Following Eight Questions. Round ALL your calculations to at least three decimal digits.

- (a) [10 marks] Let $\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq \pi, 0 \leq x_2 \leq \sin x_1\}$ and $\mathbf{y} = (\pi, 1)^t$. Given that the root of the equation $t + (\cos t) [\sin t - 1] = \pi$ is about 2.663, find the minimum distance from \mathbf{y} to \mathbb{S} , the unique minimizing point, and a separating hyperplane using the First Separation Theorem.
 - (b) [10 marks] Are the first-order and second-order Taylor approximations of $f(x_1, x_2) = \ln(x_1 + 1) - 3x_2 + 5$ about the point $(0, 1)^t$ convex, concave, or neither? Explain.
2. Consider the following problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & 2e^{x_1} - 2x_1x_2 \\ \text{s.t.} \quad & 3x_1 + 3x_2 - 6 = 0. \end{aligned}$$

- (a) [10 marks] Formulate a suitable penalty problem that can be solved using a second-order optimization method.
 - (b) [10 marks] Perform one iteration of Newton's method for solving the penalty problem obtained in Part (a) using the penalty parameter $\mu = 10$ and starting from the point $(1, 1)^t$.
3. (a) [10 marks] Let $f, g_i, h_j : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m; j = 1, \dots, l$ be continuous functions, and consider the following primal problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m, \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l, \\ & \mathbf{x} \in \mathbb{X} \end{aligned}$$

where $\mathbb{X} \in \mathbb{R}^n$ is a nonempty, compact set. Let θ be the Lagrangian dual function defined by $\theta(\mathbf{u}, \mathbf{v}) = \inf_{\mathbf{x} \in \mathbb{X}} \left\{ f(\mathbf{x}) + \sum_{i=1}^m u_i g_i(\mathbf{x}) + \sum_{j=1}^l v_j h_j(\mathbf{x}) \right\}$, where $\mathbf{u} = (u_1, \dots, u_m)^t \geq \mathbf{0}$ and $\mathbf{v} = (v_1, \dots, v_l)^t$. If $\bar{\xi}$ is the subgradient in $\partial\theta(\mathbf{u}, \mathbf{v})$ having the smallest Euclidean norm, where $\partial\theta(\mathbf{u}, \mathbf{v})$ is the subdifferential of θ at (\mathbf{u}, \mathbf{v}) , then show that the direction of steepest ascent of θ at (\mathbf{u}, \mathbf{v}) is given by

$$\bar{\mathbf{d}} = \begin{cases} \mathbf{0}, & \bar{\xi} = \mathbf{0}, \\ \frac{\bar{\xi}}{\|\bar{\xi}\|}, & \bar{\xi} \neq \mathbf{0}. \end{cases}$$

(b) **[10 marks]** Find and identify the minimum point(s) of the function

$$f(x_1, x_2) = 2x_1^3 + x_2^3 - 2x_1^2 + x_2^2 + 1.$$

4. (a) **[10 marks]** Let $f, g_i, h_j : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m; j = 1, \dots, l$, and consider the following primal problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m, \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l, \\ & \mathbf{x} \in \mathbb{X} \end{aligned}$$

where \mathbb{X} is a nonempty, open set in \mathbb{R}^n . Let $\mathbf{u} = (u_1, \dots, u_m)^t$ and $\mathbf{v} = (v_1, \dots, v_l)^t$, and define

$$\mathcal{X}(\mathbf{u}, \mathbf{v}) = \left\{ \bar{\mathbf{x}} : \bar{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{X}} \left[f(\mathbf{x}) + \sum_{i=1}^m u_i g_i(\mathbf{x}) + \sum_{j=1}^l v_j h_j(\mathbf{x}) \right] \right\},$$

to be the set of optimal solutions to the Lagrangian dual subproblem. If $|g_i(\hat{\mathbf{x}})| \leq \epsilon, \forall i \in \mathcal{I}, g_i(\hat{\mathbf{x}}) \leq \epsilon \forall i \notin \mathcal{I}$, and $|h_j(\hat{\mathbf{x}})| \leq \epsilon, j = 1, \dots, l$, at a point $\hat{\mathbf{x}} \in \mathcal{X}(\hat{\mathbf{u}}, \hat{\mathbf{v}})$, where $\hat{\mathbf{u}} \in \mathbb{R}^m : \hat{\mathbf{u}} \geq \mathbf{0}, \hat{\mathbf{v}} \in \mathbb{R}^l, \mathcal{I} = \{i : u_i > 0\}$, and ϵ is a sufficiently small positive number, then show that $\hat{\mathbf{x}}$ is a near optimal primal solution.

(b) **[10 marks]** Perform two iterations of the steepest descent method for solving the minimization problem $\min_{x_1, x_2} (2x_1^2 + (x_2 - 1)^2 - 2x_1x_2)$ starting from the point $\mathbf{x}_1 = \mathbf{0}$.

5. Consider the following problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & 2x_1^2 + x_2^2 - 2x_1x_2 \\ \text{s.t.} \quad & -2x_1 + 3x_2 \leq 6, \\ & 2x_1 + 4x_2 \geq 4, \\ & x_1, x_2 \geq 1, \\ & x_1, x_2 \leq 6. \end{aligned}$$

(a) **[10 marks]** Find the exact solution of the problem.

(b) **[10 marks]** Suppose we want to minimize the same objective function under the same constraints, then would there be a duality gap between the minimization problem and the Lagrangian dual problem obtained by letting the domain $\mathbb{X} = \{(x_1, x_2)^t : x_1, x_2 \geq 1, x_1, x_2 \leq 6\}$? Explain.

6. Consider the cutting plane algorithm for solving a Lagrangian dual problem of the following minimization problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & 2x_1^2 + x_2^2 \\ \text{s.t.} \quad & -2x_1 + 3x_2 \leq 6, \\ & x_1, x_2 \geq 1. \end{aligned}$$

- (a) **[14 marks]** If the Lagrangian dual problem is obtained by letting the domain $\mathbb{X} = \{(x_1, x_2)^t : x_1, x_2 \geq 1\}$, perform two iterations of the cutting plane algorithm for solving the dual problem using the starting point $\mathbf{x}_0 = (2, 1)^t$.
- (b) **[6 marks]** Are the approximations obtained in the second iteration optimal? Explain.
7. Consider the problem of minimizing x_1x_2 subject to $x_1 + x_2 \geq 2$ and $x_2 \geq x_1$.
- (a) **[14 marks]** Find a point satisfying the Karush-Kuhn-Tucker conditions.
- (b) **[6 marks]** Is the point obtained in Part (a) an optimal solution.
8. (a) **[10 marks]** Consider the following problem:

$$\min_{x_1, x_2} (3 - x_1 - x_2)^2 + (x_2^2 - x_1)$$

Perform one complete iteration of Davidon-Fletcher-Powell method for solving the problem starting from the point $\mathbf{x}_1 = \mathbf{0}$.

- (b) Let \mathbf{g} and \mathbf{h} be the vector functions whose components are g_1, \dots, g_m and h_1, \dots, h_l , respectively, and consider the following two problems:

$$\begin{array}{ll} \min_{\mathbf{x}} & f(\mathbf{x}) \\ \text{(Problem 1)} & \text{s.t. } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \\ & \mathbf{h}(\mathbf{x}) = \mathbf{0}, \\ & \mathbf{x} \in \mathbb{X} \end{array} \qquad \begin{array}{ll} \sup & \theta(\mu) \\ \text{(Problem 2)} & \text{subject to } \mu \geq 0 \end{array}$$

where $f, g_1, \dots, g_m, h_1, \dots, h_l$ are continuous functions defined on $\mathbb{R}^n, \mathbb{X} \subseteq \mathbb{R}^n$ is a nonempty set, $\theta(\mu) = \inf\{f(\mathbf{x}) + \mu\alpha(\mathbf{x}) : \mathbf{x} \in \mathbb{X}\}$ such that α is defined by

$$\alpha(\mathbf{x}) = \sum_{i=1}^m \phi[g_i(\mathbf{x})] + \sum_{i=1}^l \psi[h_i(\mathbf{x})],$$

and ϕ and ψ are continuous functions satisfying the following:

$$\begin{array}{ll} \phi(y) = 0 & \text{if } y \leq 0 \quad \text{and} \quad \phi(y) > 0 \quad \text{if } y > 0, \\ \psi(y) = 0 & \text{if } y = 0 \quad \text{and} \quad \psi(y) > 0 \quad \text{if } y \neq 0. \end{array}$$

Suppose also that $\exists \mathbf{x}_\mu \in \mathbb{X} : \theta(\mu) = f(\mathbf{x}_\mu) + \mu\alpha(\mathbf{x}_\mu), \quad \forall \mu \geq 0$. Prove that the following two statements hold true:

- i. **[4 marks]** $\inf\{f(\mathbf{x}) : \mathbf{x} \in \mathbb{X}, \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}\} \geq \sup_{\mu \geq 0} \theta(\mu)$.
- ii. **[6 marks]** $f(\mathbf{x}_\mu)$ is a nondecreasing function of $\mu, \theta(\mu)$ is a nondecreasing function of μ , and $\alpha(\mathbf{x}_\mu)$ is a nonincreasing function of μ .