King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 667 Final Exam
The First Semester of 2021-2022 (211)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: 1.) Write the following equations as a quasi-linear system
(5pts)

$$u^2 + \frac{x}{2}\frac{\partial u^2}{\partial x} + u\frac{\partial v}{\partial x} - vx\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v^2}{\partial y} = 0,$$

$$\frac{1}{2}\frac{\partial u^2}{\partial x} - u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v^2}{\partial y} = 0.$$

(**Do not solve it**).

2.) Solve the quasilinear system that solution passes through the curve Γ: $u=-1$, $v=1$ on $y=x$.
(15pts)

$$u\frac{\partial}{\partial x}\begin{pmatrix}u\\ v\end{pmatrix} - v\frac{\partial}{\partial y}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}\frac{u+1}{t+2}\\ v^2\end{pmatrix}.$$

<u>**Solution:**</u>

1.)
$$u\left(u + x\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) - v\left(x\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$$

$$u\frac{\partial}{\partial x}(xu+v) - v\frac{\partial}{\partial y}(xu+v) = 0$$

and
$$u\frac{\partial}{\partial x}(u-v) - v\frac{\partial}{\partial y}(u-v) = 0$$

$$\Rightarrow \boxed{u\frac{\partial}{\partial x}\begin{pmatrix}xu+v\\ u-v\end{pmatrix} - v\frac{\partial}{\partial y}\begin{pmatrix}xu+v\\ u-v\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}}$$

2.)
$$\frac{dx}{dt} = u \;;\; \frac{dy}{dt} = -v \;;\; \frac{du}{dt} = \frac{u+1}{t+2} \;;\; \frac{dv}{dt} = v^2$$

$$\int\frac{du}{u+1} = \int\frac{dt}{t+2} \qquad\qquad \int\frac{dv}{v^2} = \int dt$$

$$\ln|u+1| = \ln|t+2| + C \qquad\qquad -\frac{1}{v} = t + C_2$$

$$u+1 = C_1(t+2) \qquad\qquad v = -\frac{1}{t+C_2}$$

Thus, $\frac{dx}{dt} = -1 + C_1(t+2)$

$$x = -t + C_1\frac{t^2}{2} + 2C_1 t + C_3$$

$$\frac{dy}{dt} = \frac{1}{t+C_2} \Rightarrow y = \int\frac{dt}{t+C_2} = \ln|t+C_2| + C$$

$$\Rightarrow e^y = C_4(t+C_2).$$

At $t=0$, $s = C_3$; $e^s = C_4\cdot C_2$; $0 = 2C_1 \Rightarrow C_1 = 0$

$1 = -\frac{1}{C_2} \Rightarrow C_2 = -1$. Thus, $C_4 = -e^s$

$$\Rightarrow u=-1 \;;\; v = -\frac{1}{t-1} \;;\; x = -t+s \;;\; e^y = -e^s(t-1)$$

$$\boxed{u=-1 \text{ and } ve^y = e^{x+1-\frac{1}{v}}}$$

Problem 2: 1.) Consider the second order PDE
(5pts)

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 1, \tag{0.4}$$

Write this equation as a first order system of PDEs. **Do not solve it.**
2.) Solve the Cauchy problem
(15pts)

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + 2\dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial v}{\partial x} + 2\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} = 0, \\ u = x; \quad v = 1 \text{ on } y = -x. \end{cases} \tag{0.5}$$

<u>**Solution:**</u>

1.) Let $v = \frac{\partial u}{\partial x}$ and $w = \frac{\partial u}{\partial y}$. We have $\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}$

$$\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial w}{\partial y} + \frac{\partial v}{\partial y} = 1$$

$$\Rightarrow \frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2.) $\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{A}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. A has two distincts eigenvalues $\lambda = -1, 3$

$(A+I)k_1 = 0 \Rightarrow k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $(A-3I)k_2 = 0 \Rightarrow k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus, $\frac{\partial}{\partial x}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \tilde{u}_y \\ \tilde{v}_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\Leftrightarrow \begin{cases} \dfrac{\partial \tilde{u}}{\partial x} - \dfrac{\partial \tilde{u}}{\partial y} = 0 & (1) \\ \tilde{u}(x, -x) = \dfrac{x-1}{2} \end{cases} ; \quad \begin{cases} \dfrac{\partial \tilde{v}}{\partial x} + 3\dfrac{\partial \tilde{v}}{\partial y} = 0 & (2) \\ \tilde{v}(x, -x) = \cancel{\dfrac{x+1}{2}} \end{cases}$$

P(1): $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = -1 \Rightarrow y = -x$ is a characteristic, $\frac{d\tilde{u}}{dt} = 0$

P(2) $\frac{dx}{dt} = 1$, $\frac{dy}{dx} = 3 \Rightarrow y = 3x$ is a characteristic, $\frac{d\tilde{v}}{dt} = 0$

On $y = -x$, $\frac{\partial}{\partial x} = -\frac{\partial}{\partial y}$ $\Rightarrow \begin{cases} \dfrac{\partial \tilde{u}}{\partial x} = 0 \Rightarrow \tilde{u}(x, -x) = c \\ \dfrac{\partial \tilde{v}}{\partial x} = 0 \Rightarrow \tilde{v}(x, -x) = c. \end{cases}$

This is incompatible with the initial conditions
Thus, the Cauchy problem has no solution. 3

Problem 3: Solve the Cauchy problem
(20 pts)

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} - 4\dfrac{\partial v}{\partial y} = x, \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} - 3\dfrac{\partial v}{\partial y} = 0, \\ u = 1,\ v = 0 \text{ on } y = x. \end{cases} \tag{0.6}$$

Solution:

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}}_{A}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$ A has a repeated eigenvalue $\lambda = -1, -1$.

$(A+I)K_1 = 0$, $K_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $(A+I)K_2 = K_1$, $K_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$; $T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

$\Rightarrow \dfrac{\partial}{\partial x}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \tilde{u}_y \\ \tilde{v}_y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$; $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = T^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$; $\begin{pmatrix} \tilde{u}(x,x) \\ \tilde{v}(x,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We solve the two following problems

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial x} - \dfrac{\partial \tilde{u}}{\partial y} + \dfrac{\partial \tilde{v}}{\partial y} = 0 \\ \tilde{u}(x,x) = 0 \end{cases} \quad (1); \quad \begin{cases} \dfrac{\partial \tilde{v}}{\partial x} - \dfrac{\partial \tilde{v}}{\partial y} = x \\ \tilde{v}(x,x) = 1 \end{cases} \quad (2)$$

First, we solve (2).

$\dfrac{dx}{dt} = 1$; $\dfrac{dy}{dt} = -1$; $\dfrac{d\tilde{v}}{dt} = x \Rightarrow x = t + c_1$; $y = -t + c_2$; $\tilde{v} = \dfrac{t^2}{2} + c_1 t + c_3$

At $t=0$, $s = c_1$, $s = c_2$; $1 = c_3 \Rightarrow x = t+s$; $y = -t+s$; $\tilde{v} = \dfrac{t^2}{2} + st + 1$ $\Rightarrow s = \dfrac{x+y}{2}$, $t = \dfrac{x-y}{2}$

$\Rightarrow \tilde{v}(x,y) = \frac{1}{2}(x^2 - xy + 2)$.

Next, we solve (1)

$\dfrac{dx}{dt} = 1$; $\dfrac{dy}{dt} = -1$, $\dfrac{d\tilde{u}}{dt} = -\dfrac{\partial \tilde{v}}{\partial y} = \dfrac{x}{2}$.

$\Rightarrow x = t + c_1$; $y = -t + c_2$, $\tilde{u} = \dfrac{t^2}{4} + \dfrac{c_1}{2}t + c_3$

At $t=0$, $s = c_1$; $s = c_2$, $0 = c_3 \Rightarrow \tilde{u} = \dfrac{t^2}{4} + \dfrac{s}{2}t$; $\begin{array}{l} x = t+s \\ y = -t+s \end{array}$

$\Rightarrow \tilde{u}(x,y) = \frac{1}{16}(3x^2 - y^2 - 2xy)$

Finally, $\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = T\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 2\tilde{u} + \tilde{v} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} \frac{1}{8}(7x^2 - 6xy - y^2 + 8) \\ \frac{1}{16}(3x^2 - y^2 - 2xy) \end{pmatrix}$

Problem 4: Covert the following problem into solving a Laplace problem
(20 pts)

$$\begin{cases} \dfrac{\partial u}{\partial x} + 4\dfrac{\partial u}{\partial y} + 2\dfrac{\partial v}{\partial y} = x, \\ \dfrac{\partial v}{\partial x} - 5\dfrac{\partial u}{\partial y} + 6\dfrac{\partial v}{\partial y} = y, \end{cases} \tag{0.7}$$

(**do not solve** the obtained Laplace problem).

<u>Solution:</u>

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 4 & 2 \\ -5 & 6 \end{pmatrix}}_{A}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

A has complex eigenvalues $\lambda = \pm 3$

Let $\begin{cases} x = \frac{1}{3}\eta \\ y = \xi + \frac{5}{3}\eta \end{cases} \Rightarrow \begin{cases} \xi = -5x + y \\ \eta = 3x \end{cases} \Rightarrow \begin{cases} \partial_x = -5\partial_\xi + 3\partial_\eta \\ \partial_y = \partial_\xi \end{cases}$

$$\Rightarrow \begin{aligned} &-5u_\xi + 3u_\eta + 4u_\xi + 2V_\xi = \frac{\eta}{3} \Rightarrow -u_\xi + 3u_\eta + 2V_\xi = \frac{\eta}{3} \\ &-5V_\xi + 3V_\eta - 5u_\xi + 6V_\xi = \xi + \frac{5}{3}\eta \Rightarrow V_\xi + 3V_\eta - 5u_\xi = \xi + \frac{5}{3}\eta \end{aligned}$$

$$\Rightarrow \begin{cases} u_\eta - \frac{1}{3}u_\xi + \frac{2}{3}V_\xi = \frac{\eta}{9} \\ V_\eta - \frac{5}{3}u_\xi + \frac{1}{3}V_\xi = \frac{\xi}{3} + \frac{5}{9}\eta \end{cases} \Rightarrow \begin{pmatrix} u_\eta \\ V_\eta \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}}_{B}\begin{pmatrix} u_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} \eta/9 \\ \frac{\xi}{3} + \frac{5}{9}\eta \end{pmatrix}$$

B has the complex eigenvalues $\lambda = \pm i$

$(A - iI)K = 0 \Rightarrow K\begin{pmatrix} 1 \\ \frac{1}{2} + \frac{3}{2}i \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + i\begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$; $T = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 3/2 \end{pmatrix}$

Thus, $\begin{pmatrix} \tilde{u}_\eta \\ \tilde{V}_\eta \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{u}_\xi \\ \tilde{V}_\xi \end{pmatrix} = T^{-1}\begin{pmatrix} \eta/9 \\ \frac{\xi}{3} + \frac{5}{9}\eta \end{pmatrix} = \begin{pmatrix} \eta/9 \\ \frac{2}{9}\xi + \frac{\eta}{3} \end{pmatrix}$

$$\Rightarrow \begin{aligned} \tilde{u}_\eta + \tilde{V}_\xi &= \eta/9 \Rightarrow \tilde{u}_{\eta\eta} + \tilde{V}_{\xi\eta} = \frac{1}{9} \\ \tilde{V}_\eta - \tilde{u}_\xi &= \frac{2}{9}\xi + \frac{\eta}{3} \Rightarrow \tilde{V}_{\eta\xi} - \tilde{u}_{\xi\xi} = \frac{2}{9} \end{aligned} \Rightarrow \tilde{u}_{\eta\eta} + \tilde{u}_{\xi\xi} = \frac{-1}{9}$$

We also have

$$\begin{aligned} \tilde{u}_{\eta\xi} + \tilde{V}_{\xi\xi} &= 0 \\ \tilde{V}_{\eta\eta} - \tilde{u}_{\eta\xi} &= \frac{1}{3} \end{aligned} \Rightarrow \tilde{V}_{\eta\eta} + \tilde{V}_{\xi\xi} = \frac{1}{3}$$

(5)

Problem 5: Let Ω an open bounded domain of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$.

1.) Consider the Dirichlet problems
(5pts)

$$-\Delta u = 1, \quad (x, y, z) \in \Omega, \tag{0.8}$$
$$u|_{\partial\Omega} = 0. \tag{0.9}$$

Give a function $r = r(x, y, z)$ such that $v = r + u$ satisfies the problem

$$-\Delta v = 0, \quad (x, y, z) \in \Omega, \tag{0.10}$$
$$v|_{\partial\Omega} = g, \tag{0.11}$$

where g is continuous.

2.) Based on the potential theory, we can construct a solution $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of (0.10)-(0.10) of the form

$$v(X) = -\iint_{\partial\Omega} h(Y) \frac{\partial G}{\partial n}(X-Y) d\sigma_Y \tag{0.12}$$

where σ_Y is the surface area, $Y \in \partial\Omega$, $G(X) = \frac{1}{4\pi|X|}$ is the fundamental solution of (0.10) but for $(x, y, z) \in \mathbb{R}^3$.

(5 pts) **a.)** Explain how the expression of G is obtained?

(10 pts) **b.)** Explain how h has to be chosen.

Solution:

1.) $-\Delta V = \underbrace{-\Delta u}_{=1} - \Delta r = 0 \Rightarrow \Delta r = 1$

So, we can take $r = \frac{x^2}{2}$ or $\frac{x^2+y^2}{4}$, or $\frac{x^2+y^2+z^2}{6}$

2.) a) We set $r = |X| = \sqrt{x^2+y^2+z^2}$, $X = (x, y, z)$

$V(X) = \tilde{V}(r) \Rightarrow \frac{\partial V}{\partial x} = \frac{\partial \tilde{V}}{\partial r}\frac{\partial r}{\partial x}$

$\frac{\partial^2 V}{\partial x^2} = \tilde{V}_{rr}\frac{x^2}{r^2} - \tilde{V}_r \frac{x^2}{r^3} + \frac{1}{r}\tilde{V}_r$

$-\Delta V = 0 \Rightarrow \frac{2}{r}\tilde{V}_r + \tilde{V}_{rr} = 0 \Rightarrow \tilde{V}(r) = \frac{b}{r} + c,\ r > 0$

We also have $-\Delta G = \delta_0$ in $\mathbb{R}^3$. $\Rightarrow c = 0, b = \frac{1}{4\pi}$
(Dirac measure)

b) $g(X_0) = \frac{1}{2} h(X_0) - \iint_{\partial\Omega} h(Y) \frac{\partial G}{\partial n}(X_0 - Y)\, d\sigma_Y$