STAT 416- Major Exam 1

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1 Exercise 1(20=4+3+5+3+5 points)

Consider the following Markov chain with states $\{0,1,2,3,4\}$ and transition probabilities matrix given by

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Determine the classes, and specify which are recurrent or transient states.
2. Find the period of all states.
3. Find f_{30}.
4. Find π_0. Justify its existence.
5. Find $\lim_{n\to\infty} p_{30}^{(n)}$ and $\lim_{n\to\infty} p_{33}^{(n)}$.

2 Exercise 2(15=7+8 points)

The following is the transition probability matrix of a Markov chain with states $\{1,2,3,4\}$

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0.25 & 0.25 & 0.5 & 0 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{pmatrix}$$

If $X_0 = 1$

1. find the probability that state 3 is entered before state 4;
2. find the mean number of transitions until either state 3 or state 4 is entered.

3 Exercise 3(5 points)

Suppose that on each play of the game a gambler either wins 1 with probability p or loses 1 with probability $1-p$. The gambler continues betting until she or he is either up n or down m. What is the probability that the gambler quits a winner?

4 Exercise 4(10=7+3 points)

Consider a branching process having $\mu < 1$. Show that if $Z_0 = 1$, then the expected number of individuals that ever exist, that is $\sum_{n=0}^{\infty} Z_n$, in this population is given by $1/(1-\mu)$. What if $Z_0 = n$?

5 Exercise 5(15=5+5+5 points)

The state of a process changes daily according to a two-state Markov chain. If the process is in state i during one day, then it is in state j the following day with probability $p_{i,j}$, where $p_{0,0} = 0.4$, $p_{0,1} = 0.6$, $p_{1,0} = 0.2$ and $p_{1,1} = 0.8$. Every day a message is sent. If the state of the Markov chain that day is i then the message sent is good with probability p_i and is bad with probability $q_i = 1 - p_i$, $i = 0, 1$

1. If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
2. If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
3. In the long run, what proportion of messages are good?

6 Exercise 6(10=3+7 points)

Suppose that a Markov chain has two states $\{0, 1\}$ where

$$P^{(n)} = \frac{1}{2} \begin{pmatrix} 1 + (2p-1)^n & 1 - (2p-1)^n \\ 1 - (2p-1)^n & 1 + (2p-1)^n \end{pmatrix}$$

for $n \geq 1$.

1. Write the transition matrix.
2. If $P(X_0 = 0) = 1/4$ and $P(X_0 = 1) = 3/4$, calculate $E(X_n)$ for $n \geq 1$.