STAT 416- Final Exam

KFUPM, Department of Mathematics and Statistics

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Instructions: You must show all your work. No materials are allowed except a calculator.

1 Exercise 1(2+2+2+2+2 points)

1. What is the definition of λ_a in the queuing theory?
2. Give the formula of r_i: the average arrival rate at station i
3. Define GI/G/1 queue.
4. For an M/G/1 queue, where the mean of the service time is μ and the arrival rate is λ, give the average length of a busy period.
5. Define an alternating renewal process.

2 Exercise 2(3+7+5+2 points)

Customers arrive at a two-server station according to a Poisson process with rate λ. Upon arriving they join a single queue to wait for the next available server. Suppose that the service times of the two servers a and b are exponential with the same rate μ and that a customer who arrives to find the system empty will go to server a. Suppose that $\lambda < 2\mu$.

1. Formulate a Markov chain model for this system. Find the different rates $q_{i,j}$? Give v_i for any i?
2. Find the stationary distribution of this Markov chain.
3. On average, how many customers are in the system?
4. Find the average time a customer spends in the system.

3 Exercise 3 (8+2 points)

A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 10. The amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8.

1. What is the average number of machines not in use?
2. What proportion of time are both repairman busy?

4 Exercise 4(4+4+3+3 points)

Consider a three station queuing network in which arrivals to servers $i = 1, 2, 3$ occur at rates $3, 2, 1$, respectively. Service at stations $i = 1, 2, 3$ occurs at rates $4, 5, 6$, respectively. Suppose that the probability of going to station j when exiting station i is given by $p_{1,2} = 1/3$, $p_{1,3} = 1/3$, $p_{2,3} = 2/3$, and $p_{i,j} = 0$ otherwise.

1. Describe the network by the its transition graph.
2. Show that the system is stable.
3. Find its stationary distribution
4. Find the average number of customers in the system.

5 Exercise 5(10+8 points)

Suppose that the lifetime of a car is a random variable with density f. Our methodical Mr. Brown buys a new car as soon as the old one breaks down or reaches T years. Suppose that a new car costs A dollars and that an additional cost of B dollars to repair the vehicle is incurred if it breaks down before time T.

1. What is the long-run cost per unit time of Mr. Brown's policy?
2. Suppose now that $f(t) = \frac{e^{-t/4}}{4}$, $T = 5$, $A = 10$ and $B = 1.5$. What is this long-run cost per unit time.

6 Exercise 6(8+3 points)

A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60; each time he drives from B to A, he drives at a fixed speed that is equally likely to be either 40 or 60.

1. In the long run, what proportion of his driving time is spent going to B?
2. In the long run, for what proportion of his driving time is he driving at a speed of 40 miles per hour

7 Exercise 7(10+5+5 points)

There are three machines, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i = 1, 2, 3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is exponential with rate 5; the time to fix machine 2 is uniform on $(0, 4)$; and the time to fix machine 3 is a gamma random variable with parameters $n = 3$ and $\lambda = 2$. Once a failed machine is repaired, it is as good as new and all machines are restarted.

1. What proportion of time is the system working?

2. What proportion of time is machine 1 being repaired?

3. What proportion of time is machine 2 in a state of suspended animation (that is, neither working nor being repaired)?

Formula sheet

1.
$$\sum_{n=m}^{\infty} x^n = \frac{x^m}{1-x},$$
$$\sum_{n=m}^{\infty} nx^n = \frac{x^m(m+x-mx)}{(1-x)^2},$$
for any integer $m \geq 0$.

2.
$$\pi_n = \frac{\theta_n}{\sum_{n=0}^{\infty} \theta_n},$$
for $n = 0, 1, 2, \ldots$, where $\theta_0 = 1$ and $\theta_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}$ for $n \geq 1$.

3.
$$\int xe^{ax}dx = \frac{(ax-1)e^{ax}}{a^2} + C.$$

4. Little's formulas $L = \lambda_a W$ and $L_Q = \lambda_a W_Q$.

5. $L_Q = L - 1 + \pi(0)$.

6. $E[Gamma(n, \lambda)] = n/\lambda$.

7. For a network of k stations, we have
   (a) $\pi(n_1, n_2, \ldots, n_k) = \prod_{i=1}^{k} \left(1 - \frac{r_i}{\mu_i}\right) \left(\frac{r_i}{\mu_i}\right)^{n_i}$.
   (b) $L = \sum_{i=1}^{k} \frac{r_i}{\mu_i - r_i}$.